Competitive Equilibrium in Asset Markets with Adverse Selection*

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Abstract

We develop a theory of equilibrium in asset markets with adverse selection. Traders can buy and sell an asset at any price. Sellers recognize that their trades may be rationed if they ask for a high price, while buyers recognize that they can only get a high quality good by paying a high price. These beliefs are consistent with rational behavior by the traders on the other side of the market. In the resulting equilibrium, the existence of low-quality assets reduces the liquidity and price-dividend ratio in the market for high quality assets. A larger player who purchases and destroys all the low quality assets will improve the liquidity and raise the price-dividend ratio for the remaining assets.

1 Introduction

This paper develops a dynamic equilibrium model of asset markets with adverse selection. Sellers can attempt to sell a durable asset at any price. Buyers must form rational expectations about the type of asset that is available at each price. In equilibrium, sellers are rationed by a shortage of buyers at all prices except the lowest one, and it is increasingly difficult to sell an asset at higher prices. This keeps the owners of low quality assets from trying to sell them at high prices. On the other hand, the owners of high quality assets are willing to set a high price despite the low sale probability because the asset is worth more to them if they fail to sell it.

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Our model offers an abstract view of an illiquid asset market, for example the market for asset-backed securities during the 2008 financial crisis. Prior to the crisis, market participants viewed AAA securities as a safe investment, indistinguishable from Treasuries; indeed, they were treated as such by banking regulators. In the early stages of the crisis, investors started to recognize that some of these securities were likely to pay less than face value. Moreover, it was difficult to determine the exact assets that backed each individual security. Anticipating that she might later have to sell it, at this point it started to pay for the owner of an asset to learn its quality. On the other hand, it did not pay potential buyers to investigate the quality of all possible assets because they did not know which assets would later be for sale. This created an adverse selection problem, where sellers have superior information than buyers, as in the classic market for lemons (Akerlof, 1970).

We predict that within an asset class, such as AAA-rated mortgage backed securities, a seller should always be able to sell an asset at a sufficiently low price. However, the owners of good quality assets will choose to hold out for a higher price, recognizing that there will be a shortage of buyers at that price and so it will take time sell the asset. Moreover, the price that buyers are willing to pay for a high quality asset will be depressed because the market is less liquid. That is, even if a buyer somehow understood that a particular mortgage-backed security would pay the promised dividends with certainty, he would pay less for it than for a Treasury because he would anticipate having trouble reselling the MBS to future buyers who don't have his information. Illiquidity therefore serves to further depress asset prices. In particular, the ability of sellers to learn the quality of their assets will depress the liquidity and may depress the value of all securities even if the average the quality is unchanged.

An obvious solution to this problem is to have a third party evaluate the quality of the assets. Indeed, this is the role that the rating agencies were supposed to play. But the rating agencies lost their credibility during the crisis and there was no one with the reputation and capability to take their place. We find instead that there may have been a role for an investor with deep pockets, such as a government, to purchase low quality assets and alleviate the illiquidity of high quality ones. In particular, suppose the government stood ready to buy all assets at a moderate price. Any asset which the seller believed was worth less than that price, even if fully liquid, would be sold to the government, which in turn would take a loss on its purchases. The elimination of trade in low quality assets moderates the adverse selection problem. This makes all other assets more liquid and more expensive. Thus asset purchases can potentially alleviate both illiquidity and insolvency.

Our model is deliberately stylized. Assets are perfectly durable and pay a constant dividend, a nondurable consumption good. Better quality assets pay a higher dividend. Consumers are risk-neutral and have a discount factor that shifts randomly over time, creating a reason for trade. The only permissible trades are between the consumption good and the asset. Still, we believe this framework is useful for capturing our main idea that illiquidity may serve to separate high and low quality assets. In particular, it is a dynamic general equilibrium model in which the distribution of asset holdings evolves endogenously over time as consumers trade and experience preference shocks. We define a competitive equilibrium in this environment and prove that it is unique. In equilibrium, higher quality assets trade at a higher price but with a lower probability. Indeed, the expected revenue from selling a an asset, the product of its price and trading probability, is decreasing in the quality of the asset.

We also show that the trading frictions in this environment do not depend on any assumptions about the frequency of trading opportunities. Even with continuous trading opportunities, there are not enough buyers in the market for high quality assets and so it takes a real amount of calendar time to sell at a high price. This is in contrast to models that emphasize illiquidity in asset markets due to search frictions, such as Duffie, Gârleanu and Pedersen (2005), Weill (2008), and Lagos and Rocheteau (2009), where the economy converges to the frictionless outcome when the time between trading opportunities goes to zero. In our adverse selection economy, real trading delays are essential for separating the good assets from the bad ones. Of course, in reality adverse selection and search frictions may coexist in a market, and it is indeed straightforward to introduce search into our framework Guerrieri, Shimer and Wright (2010); Chang (2010).

There is a large literature on dynamic adverse selection models. In many cases, the authors implicitly assume that all trades must take place at one price, so there is necessarily a pooling equilibrium (Eisfeldt, 2004; Kurlat, 2009, e.g.). This implies that sellers choose not to sell some assets because the price is too low, and so in a sense these models also deliver illiquidity. In our model, in contrast, sellers try to sell all their assets, but most only sell with some probability between 0 and 1. Moreover, our model allows for the possibility that a seller can demand a high price for her asset, something that models which impose a uniform price cannot address. For an alternative approach to dynamic adverse selection, see also Daley and Green (2010) and the references therein.

This paper builds on our previous work with Randall Wright, Guerrieri, Shimer and Wright (2010). It also complements a contemporaneous paper by Chang (2010). There are a number of small differences between that paper and this one. For example, we look at an environment in which consumers may later want to resell assets that they purchase today. This means that buyers care about the liquidity of the asset. We allow consumers to hold multiple assets, although that turns out to be inessential for our analysis. We also focus more explicitly on general equilibrium, allowing for the possibility that buyers may be driven to a corner in which they do not consume anything. Still, both papers leverage our earlier research to study separating equilibria in a dynamic adverse selection environment.

This paper proceeds as follows. Section 2 describes our basic model. Section 3 describes the consumer's problem and shows how to express it recursively. Section 4 defines equilibrium and establishes existence and uniqueness. Section 5 provides closed-form solutions for a version of the model with a continuum of assets. Section 6 extends the model to have persistent preference shocks and then shows that the frictions survive in the continuous time limit. Section 7 concludes with a brief recap of how our model can generate fire sales following the revelation of some information and how illiquidity and insolvency can be alleviated through an asset purchase program, although the program necessarily loses money.

2 Model

There is a unit measure of risk-neutral consumers. In each period t, they can be in one of two states, $s_t \in \{l, h\}$, which determines their discount factor β_{s_t}. We assume $0 < \beta_l < \beta_h < 1$. The preference shock is independent across consumers and for now we assume that it is also independent over time. Thus π_s denotes the probability that a consumer is in state $s \in \{l, h\}$ in any period, and it is also the fraction of consumers who are in state s in any period. Let $\bar{\beta} = \pi_l\beta_l + \pi_h\beta_h$ denote the expected discount factor. For any particular consumer, let $s^t \equiv \{s_0, \ldots, s_t\}$ denote the history of states through period t.

There is a finite number of different types of assets, distinguished by their type $j = 1, \ldots, J$. The assumption that there is a finite number of asset types simplifies our notation but is inessential for our results, as we discuss below. Assets are perfectly durable and so their supply is fixed; let K_j denote the measure of type j assets in the economy. Each type j asset produces δ_j units of the homogeneous, nondurable consumption good each period, and so aggregate consumption $\sum_{j=1}^{J} \delta_j K_j$ is fixed. Without loss of generality, assume that higher type assets produce more of the consumption good, $0 \leq \delta_1 < \cdots < \delta_J$. We are interested in how a market economy allocates consumption across consumers. For the remainder of the paper, we refer to the assets as "trees" and the consumption good as "fruit."

The timing of events within period t is as follows:

1. each consumer i owns a vector $\{k_{i,j}\}$ of assets which produce fruit;

2. each consumer's discount factor between periods t and $t+1$ is realized;

3. consumers trade trees for fruit in a competitive market;

4. consumers consume the fruit that they hold.

We require that each individual's consumption is nonnegative in every period and we do not allow any other trades, e.g. contingent claims against shocks to the discount factor. In addition, we assume that only the owner of a tree can observe its quality, creating an adverse selection problem. The buyer of a tree may be able to infer its quality from the price at which it is sold, however. Finally, we impose that only low type consumers may sell trees and henceforth call them "sellers." For symmetry, we refer to high type consumers as buyers. This configuration is reasonable in the sense that, absent an adverse selection problem, high types would buy trees from low types transferring consumption from those with a high intertemporal marginal rate of substitution to those with a low one.

We now describe the competitive fruit market more precisely. After trees have borne fruit, a continuum of markets distinguished by a positive price $p \in \mathbb{R}_+$ open up. Each buyer may take his fruit to any market (or combination of markets), attempting to purchase trees in that market. Each seller may take his trees to any market (or combination of markets) attempting to sell trees in that market.

All consumers have rational beliefs about the ratio of buyers to sellers in all markets. Let $\Theta(p)$ denote the ratio of the amount of fruit brought by buyers to a market p, relative to the cost of purchasing the trees in that market at a price p. In other words, if $\Theta(p) < 1$, there is not enough fruit to purchase all the trees offered for sale in the market, while if $\Theta(p) > 1$, there is more than enough. A seller believes that if he brings a tree to a market p, it will sell with probability $\min\{\Theta(p), 1\}$. That is, if there are excess trees in the market, the seller believes that he will succeed in selling it only probabilistically. Likewise, a buyer who brings p units of fruit to market p believes that he will buy a tree with probability $\min\{\Theta(p)^{-1}, 1\}$. If there is excess fruit in the market, he will be rationed. A seller who is rationed keeps his tree until the following period, while a buyer who is rationed must eat his fruit.

Consumers also have rational beliefs about the types of tree sold in each market. Let $\Gamma(p) \equiv \{\gamma_j(p)\} \in \Delta^J$ denote the probability distribution over trees available for sale in the market, where Δ^J is the J-dimensional unit simplex.[1] Buyers expect that, conditional on buying a tree at a price p, it will be a type j tree with probability $\gamma_j(p)$. Buyers only learn the quality of the tree that they have purchased after giving up their fruit. They have no recourse if unsatisfied.

Although trade does not happen at every price p, the functions Θ and Γ are not arbitrary. Instead, consumers believe that if $\Theta(p) < \infty$, some seller must find it optimal to sell some type of trees at price p. In addition, if $\gamma_j(p) > 0$, sellers find it (weakly) optimal to sell type

[1] That is, $\gamma_j(p) \geq 0$ for all j and $\sum_{j=1}^{J} \gamma_j(p) = 1$.

j trees at price p. This restriction on beliefs narrows the set of equilibria by, for example, ruling out equilibria in which no one pays a high price for a tree because everyone believes that they will be unable to purchase a good quality tree at that price.

3 Consumer's Problem

A consumer can in principle trade at any positive price p. However, he may be rationed at some prices and the price may affect the quality of tree that he purchases.

For any period t, history s^{t-1}, and type $j \in \{1, \dots, J\}$, let $k_{i,j,t}(s^{t-1})$ denote consumer i's beginning-of-period t holdings of type j trees. For any period t, history s^t, type $j \in \{1, \dots, J\}$, and set $P \subset \mathbb{R}_+$, let $q_{i,j,t}(P; s^t)$ denote his net purchase in period t of type j trees at a price $p \in P$. The consumer chooses a history-contingent sequence for consumption $c_{i,t}(s^t)$ and measures of tree holdings $k_{i,j,t+1}(s^t)$ and net tree purchases $q_{i,j,t}(P; s^t)$ to maximize his expected lifetime utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{\tau=0}^{t-1} \pi_{s_\tau} \beta_{s_\tau} \right) \pi_{s_t} c_{i,t}(s^t),$$

subject to a budget constraint

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) = c_{i,t}(s^t) + \int_0^\infty \sum_{j=1}^{J} p\, q_{i,j,t}(\{p\}; s^t) dp$$

for all t and s^t, a law of motion for his tree holdings,

$$k_{i,j,t+1}(s^t) = k_{i,j,t}(s^{t-1}) + q_{i,j,t}(\mathbb{R}_+; s^t)$$

for all $j \in \{1, \dots, J\}$, and a set of constraints that depends on his discount factor.

If the consumer has a high discount factor, he is a buyer, which implies $q_{i,j,t}(P; s^t)$ is nonnegative for all $j \in \{1, \dots, J\}$ and $P \subset \mathbb{R}_+$ when $s_t = h$. In addition, he must have enough fruit to fund the purchase of these trees,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) \geq \int_0^\infty \sum_{j=1}^{J} p \max\{\Theta(p), 1\} q_{i,j,t}(\{p\}; s^t) dp.$$

If the consumer wishes to purchase q trees at a price p and $\Theta(p) > 1$, he will be rationed and so must bring $p\Theta(p)q$ fruit to the market to make this purchase. This constrains his ability to buy trees in markets with excess demand. Finally, he can only purchase type j trees at a price p if consumers are selling them at that price, that is

$$q_{i,j,t}(P;s^t) = \int_P \gamma_j(p) \left(\sum_{j'=1}^{J} q_{i,j',t}(\{p\};s^t) \right) dp$$

for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. The left hand side is the quantity of type j trees purchased at a price $p \in P$. The integrand on the right hand side is the product of quantity of trees purchased at price p and the share of those trees that are of type j.

If the consumer has a low discount factor, he is a seller, which implies $q_{i,j,t}(P;s^t)$ is nonpositive for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$ when $s_t = l$. In addition, he may not try to sell more trees than he owns:

$$k_{i,j,t}(s^{t-1}) \geq -\int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_{i,j,t}(\{p\};s^t) dp,$$

for all $j \in \{1, \ldots, J\}$. Each tree only sells with probability $\min\{\Theta(p), 1\}$ at price p, so if $\Theta(p) < 1$, a consumer must bring $\Theta(p)^{-1}$ trees to the market to sell one of them. Sellers are not restricted from selling trees in the wrong market. Instead, in equilibrium they will be induced not to do so.

Let $V^*(\{k_j\})$ be the supremum of the consumers' expected lifetime utility over feasible policies, given initial tree holding vector $\{k_j\}$. We prove in Proposition 1 that the function V^* satisfies the following functional equation:

$$V(\{k_j\}) = \pi_h W_h(\{k_j\}) + \pi_l W_l(\{k_j\}), \tag{1}$$

where

$$W_h(\{k_j\}) = \max_{\{q_j, k'_j\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_h V(\{k'_j\}) \right) \tag{2}$$

subject to $k'_j = k_j + q_j(\mathbb{R}_+)$ for all $j \in \{1, \ldots, J\}$

$$\sum_{j=1}^{J} \delta_j k_j \geq \int_0^\infty p \max\{\Theta(p), 1\} \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp,$$

$$q_j(P) = \int_P \gamma_j(p) \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+$$

$$q_j(P) \geq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+,$$

and

$$W_l(\{k_j\}) = \max_{\{q_j,k_j'\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_l V(\{k_j'\}) \right) \tag{3}$$

$$\begin{aligned} \text{subject to } & k_j' = k_j + q_j(\mathbb{R}_+) \text{ for all } j \in \{1,\ldots,J\} \\ & k_j \geq -\int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_j(\{p\}) dp \text{ for all } j \in \{1,\ldots,J\}, \\ & q_j(P) \leq 0 \text{ for all } j \in \{1,\ldots,J\} \text{ and } P \subset \mathbb{R}_+, \end{aligned}$$

From now on, we work with the recursive version of the consumers' problem.

Given the linearity of the problem, it is not surprising that the value functions are linear in tree holdings. That is, $V(\{k_j\}) = \sum_{j=1}^{J} v_j k_j$, where v_j denotes the expected value of a tree of type j next period and satisfies the functional equation

$$v_j \equiv \pi_h w_{h,j} + \pi_l w_{l,j} \text{ for all } \delta. \tag{4}$$

Similarly, the functions $w_{s,j}$ are such that $W_s(\{k_j\}) = \sum_{j=1}^{J} w_{s,j} k_j$ for $s \in \{l, h\}$. These satisfy relatively simple recursive problems. A seller solves

$$w_{l,j} = \delta_j + \beta_l v_j + \max_p \big(\min\{\Theta(p), 1\}(p - \beta_l v_j) \big). \tag{5}$$

The consumer earns a dividend δ_j from the tree and also gets p units of fruit if he manages to sell the tree at the chosen price p. Otherwise he keeps the tree until the following period. Note that an impatient consumer never strictly prefers to hold onto a tree rather than try to sell it, since he can always offer it at a high price $p > \beta_l v_l$. Of course, at such a high price, he may be unable to sell it, $\Theta(p) = 0$, in which case the outcome is the same as holding onto the tree.

For a buyer, a type j tree delivers δ_j units of fruit, each of which may either be consumed or used to purchase trees. A key result is that the value of fruit to a buyer is independent of the type of tree that produced that fruit. Define $1 + \lambda$ to be the value of a unit of fruit to a buyer. If $\lambda = 0$, the consumer finds it weakly optimal to eat the fruit, while if $\lambda > 0$ he strictly prefers to purchase trees. Then

$$w_{h,j} = \delta_j(1 + \lambda) + \beta_h v_j. \tag{6}$$

In addition, the value of a unit of fruit in excess of its consumption utility satisfies

$$\lambda \equiv \max_{p \geq 0} \left(\min\{\Theta(p)^{-1}, 1\} \left(\frac{\beta_h \sum_{j=1}^{J} \gamma_j(p) v_j}{p} - 1 \right) \right), \tag{7}$$

with $\lambda = 0$ if the maximum value is negative. The buyer uses the fruit to attempt to purchase $1/p$ trees at an optimally chosen price p. If he succeeds, with probability $\Theta(p)^{-1}$ if $\Theta(p) > 1$ and probability 1 otherwise, he enjoys the expected value of the tree next period but gives up a unit of fruit.

Proposition 1 *Let $\{v_j\}$, $\{w_{s,j}\}$, and λ be positive-valued numbers that solve the Bellman equations* (4), (5), (6), *and* (7) *for $s = l, h$. Then $V^*(\{k_j\}) \equiv \sum_{j=1}^{J} v_j k_j$ for all k.*

All proofs are in the appendix. Note that for some choices of of the functions Θ and Γ, there is no positive-valued solution to the Bellman equation. In this case, the price of trees is so low that it is possible for a consumer to obtain unbounded utility and there is no solution to the consumer's problem. Not surprisingly, this cannot be the case in equilibrium.

4 Equilibrium

4.1 Partial Equilibrium

We are now ready to define equilibrium. We do so in two steps, first focusing on a partial equilibrium where the buyer's value of fruit λ is fixed:

Definition 1 *A partial equilibrium with adverse selection for fixed $\lambda \geq 0$ is a pair of vectors $\{v_j\} \in \mathbb{R}_+^J$ and $\{w_{l,j}\} \in \mathbb{R}_+^J$, a set $\mathbb{P} \subset \mathbb{R}_+$, a function $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$, and a function $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$ satisfying the following conditions:*

1. *consistency of the value functions:*

$$v_j = \pi_h\big(\delta_j(1+\lambda) + \beta_h v_j\big) + \pi_l w_{l,j},$$
$$w_{l,j} = \delta_j + \beta_l v_j + \max\Big\{0, \max_{p \in \mathbb{P}} \Big(\min\{\Theta(p), 1\}\big(p - \beta_l v_j\big)\Big)\Big\},$$

   with $w_{l,j} = \delta_j + \beta_l v_j$ if $\mathbb{P} = \emptyset$.

2. *sellers' optimality: for all $p \in \mathbb{R}_+$ and $j \in \{1, \ldots, J\}$,*

$$w_{l,j} \geq \delta_j + \beta_l v_j + \min\{\Theta(p), 1\}\big(p - \beta_l v_j\big),$$

with equality if $\Theta(p) < \infty$ *and* $\gamma_j(p) > 0$*. Moreover, if* $p < \beta_l v_j$*, either* $\Theta(p) = \infty$ *or* $\gamma_j(p) = 0$.

3. *buyers' optimality: for all* $p \in \mathbb{R}_+$,

$$\lambda \geq \min\{\Theta(p)^{-1}, 1\} \left(\frac{\beta_h \sum_{j=1}^{J} \gamma_j(p) v_j}{p} - 1 \right),$$

with equality if $p \in \mathbb{P}$.

The first condition simply repeats the value functions (4), (5), and (6). The other two conditions mimic the definition of equilibrium in Guerrieri, Shimer and Wright (2010). Sellers' optimality requires that for every tree δ, the value of selling it at a price p cannot exceed $w_{l,j}$. Moreover, if the buyer-seller ratio at p is finite and buyers expect to be able to purchase type j trees with positive probability, then this must be a best price for selling that type of tree. Buyers' optimality requires that the value of a unit of fruit in excess of its consumption value cannot exceed λ and must be equal to λ if there is trade at that price in equilibrium.

For fixed λ, we find the partial equilibrium as the solution to a sequence of optimization problems:

$$w_{l,j} = \delta_j + \beta_l v_j + \max_{p,\theta} \big(\min\{\theta, 1\}(p - \beta_l v_j) \big) \tag{P-j}$$
$$\text{s.t. } \lambda \leq \min\{\theta^{-1}, 1\} \left(\frac{\beta_h v_j}{p} - 1 \right),$$
$$w_{l,j'} \geq \delta_{j'} + \beta_l v_{j'} + \min\{\theta, 1\}(p - \beta_l v_{j'}) \text{ for all } j' < j$$

where

$$v_j = \frac{\pi_h \delta_j (1 + \lambda) + \pi_l w_{l,j}}{1 - \pi_h \beta_h}.$$

To solve these problems, start with type 1 trees. The last constraint disappears from Problem (P-1), and so we can solve directly for v_1 and w_1, as well as the optimal policy p_1 and θ_1. Standard arguments ensure that the solution is unique if $\lambda \geq 0$. In general, for Problem (P-j), the first constraint and the constraint of excluding type $j-1$ trees binds, which determines p_j and θ_j as well as v_j and $w_{l,j}$. The following Lemma states this claim formally:

Lemma 1 *For fixed* $\lambda \in [0, \beta_h/\beta_l - 1]$*, the sequence of Problems (P-j) has a unique solution* $\{v_j\}$ *and* $\{w_{l,j}\}$*. The maximizers* $\{p_j\}$ *and* $\{\theta_j\}$ *are pinned down as follows: If* $\lambda = 0$*,* $\theta_1 \geq 1$*. If* $\lambda = \beta_h/\beta_l - 1$*,* $\theta_1 \in [0, 1]$*. Otherwise* $\theta_1 = 1$*. In any case,*

$$p_1 = \frac{\delta_1 \beta_h (1 + \pi_h \lambda)}{1 + \lambda - \beta_h (1 + \pi_h \lambda)} \text{ and } v_1 = \frac{\delta_1 (1 + \lambda)(1 + \pi_h \lambda)}{1 + \lambda - \beta_h (1 + \pi_h \lambda)}.$$

For $j \in \{2, \dots, J\}$, $\theta_j \le \theta_{j-1}$, $p_j > p_{j-1}$, and $v_j > v_{j-1}$ are defined recursively by the following system of equations:

$$p_j = \frac{\beta_h v_j}{1+\lambda},$$
$$v_j = \frac{\delta_j(1+\pi_h\lambda) + \pi_l\theta_j p_j}{1-\pi_h\beta_h - \pi_l\beta_l(1-\theta_j)},$$
$$\theta_j(p_j - \beta_l v_{j-1}) = \min\{\theta_{j-1}, 1\}(p_{j-1} - \beta_l v_{j-1}).$$

We focus on values of λ between 0 and $\beta_h/\beta_l - 1$ because these are the relevant ones for equilibrium. One can, however, also characterize the partial equilibrium for $\lambda > \beta_h/\beta_l - 1$.

Proposition 2 *Fix $\lambda \in [0, \beta_h/\beta_l - 1]$. There exists a unique partial equilibrium and that equilibrium is given by the solution to the Problems (P-j). More precisely:*

- *Existence: Take any $\{p_j\}$, $\{\theta_j\}$, $\{v_j\}$, and $\{w_{l,j}\}$ that solve the set of problems $\{$(P-j)$\}$. Then there exists a partial equilibrium $(v, w_l, \mathbb{P}, \Theta, \Gamma)$ where $\mathbb{P} = \{p_j\}$, $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, and $v = \{v_j\}$ and $w_l = \{w_{l,j}\}$.*
- *Uniqueness: Take any partial equilibrium $(v, w_l, \mathbb{P}, \Theta, \Gamma)$. If $p_j \in \mathbb{P}$, $\gamma_j(p_j) > 0$ and $\theta_j \equiv \Theta(p_j) < \infty$, then (p_j, θ_j) solves Problem (P-j).*

The proof gives a complete characterization of the partial equilibrium and proves that any allocation that does not solve Problem (P-j) is not a partial equilibrium

4.2 Competitive Equilibrium

We now turn to a full competitive equilibrium in which λ is endogenous:

Definition 2 *A competitive equilibrium with adverse selection is a number $\lambda \in \mathbb{R}_+$, a measure μ on $\mathbb{R}_+$ with support $\mathbb{P}$, a pair of vectors $\{v_j\} \in \mathbb{R}_+^J$ and $\{w_{l,j}\} \in \mathbb{R}_+^J$, a function $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$, and a function $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$ satisfying the following conditions:*

1. *$(\{v_j\}, \{w_{l,j}\}, \mathbb{P}, \Theta, \Gamma)$ is a partial equilibrium with fixed λ; and*
2. *markets clear:*
$$\pi_h \sum_{j=1}^{J} \delta_j K_j = \pi_l \int_{\mathbb{P}} \Theta(p) p \mu(p) dp.$$

A competitive equilibrium is a partial equilibrium plus a market clearing condition, which states that the value of trees brought to the market by sellers is equal to the fruit brought to market by buyers.

Proposition 3 *A competitive equilibrium* $(\lambda, v, w_l, \mu, \Theta, \Gamma)$ *exists and is unique.*

The proof (to be completed) shows that an increase in the value of fruit to a buyer λ drives down the price of type j trees, i.e. p_j such that $\gamma_j(p_j) > 0$ and $\mu_j(p_j) > 0$. In addition, it makes it more difficult for sellers of good quality trees to separate themselves from those selling bad trees, reducing $\Theta(p_j)$ as well. Indeed, in the limit when $\lambda = \beta_h/\beta_l - 1$, $\Theta(p_j) = 0$ for all $j > 1$, and so trade breaks down in all but the worst type of tree. At the opposite limit of $\lambda = 0$, buyers are indifferent about purchasing trees and so $\Theta(p_1) > 1$ and buyers are rationed. By varying λ, we find the unique value at which the market clear.

5 Continuous Types of Assets

We have assumed for notational convenience that there are only a finite number of types of trees. It is conceptually straightforward to extend our analysis to an environment with a continuum of trees. Rather than redo our entire analysis, we take the limit as the tree distribution becomes dense but atomless on some interval of the real line $[\underline{\delta}, \bar{\delta}]$.

The key to our analysis is that for a fixed value of λ, the partial equilibrium characterized in Proposition 2 depends only on the support of the tree distribution. In particular, the price and expected value of the lowest quality tree is

$$P(\underline{\delta}) = \frac{\underline{\delta}\beta_h(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)} \text{ and } v(\underline{\delta}) = \frac{\underline{\delta}(1+\lambda)(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)}.$$

In addition, since the distribution of trees is atomless, we may simply assume $\Theta(P(\underline{\delta})) = 1$. For $p < P(\underline{\delta})$, $\Theta(p) = \infty$ and $\Gamma(p)$ is defined arbitrarily. These results are unchanged from the model with a finite number of types of trees.

To analyze higher values of p, start with the condition that the seller of a type $j-1$ tree must be indifferent about representing it as a type j tree:

$$\Theta(p_j)(p_j - \beta_l v_{j-1}) = \min\{\Theta(p_{j-1}), 1\}(p_{j-1} - \beta_l v_{j-1}).$$

When the types of trees are dense, we can rewrite this as a first order condition. That is, differentiate the right hand side with respect to p_{j-1} and evaluate at $p_{j-1} = p_j$ and $v_{j-1} = v_j$:

$$\Theta'(p_j)(p_j - \beta_l v_j) + \Theta(p_j) = 0.$$

Also eliminate v_j using $p_j = \beta_h v_j/(1+\lambda)$. This gives

$$\Theta'(p)p\left(\frac{\beta_h - \beta_l(1+\lambda)}{\beta_h}\right) + \Theta(p) = 0.$$

If $\lambda = \beta_h/\beta_l - 1$, this implies $\Theta(p) = 0$ for all $p > P(\underline{\delta})$. Otherwise, solve this differential equation using the terminal condition $\Theta(P(\underline{\delta})) = 1$ to get

$$\Theta(p) = \left(\frac{P(\underline{\delta})}{p}\right)^{\frac{\beta_h}{\beta_h - \beta_l(1+\lambda)}} \tag{8}$$

for $p > P(\underline{\delta})$. Finally, we compute the type of tree offered at a price p. We do this by eliminating v_j from the Bellman equation using the expression $v_j = p_j(1+\lambda)/\beta_h$. This gives $\gamma_j(p) = 1$ if and only if $\delta_j = D(p)$ where

$$D(p) = p\left(\frac{1+\lambda+(\beta_h-(1+\lambda)\beta_l)(1-\Theta(p))(1-\pi_h)}{\beta_h(1+\lambda\pi_h)} - 1\right), \tag{9}$$

so buyers anticipate buying type $D(p)$ trees (and only type $D(p)$ trees) at a price p.

These equations hold as long as $\bar{\delta} \geq D(p)$. For higher prices, $\Theta(p)$ is pinned down by the indifference curve of the seller of the best type of tree.

One can of course also define a competitive equilibrium with adverse selection in this environment, so market clearing determines λ. In this case, one can directly directly use the functional form of Θ to prove that the equilibrium is unique.

6 Persistent Shocks and Continuous Time

Our model explains how adverse selection can generate liquidity frictions, in the sense that a tree only sells with a certain probability each period. But suppose that the time between periods is negligible. Will the trading frictions become negligible as well? We argue in this section that they will not. Instead, a separating equilibrium requires a real amount of calendar time before a high quality tree is sold.

To address this concern, we consider the limiting behavior of the economy when the number of periods per unit of calendar time becomes very large. That is, we take the limit as the discount factors converge to 1, holding fixed the ratio of discount rates $(1-\beta_h)/(1-\beta_l)$. But as we take this limit, we also want to avoid changing the stochastic process of shocks. With i.i.d. shocks and very short time periods, there is almost no difference in preferences between high and low types of consumers and so the gains from trade become negligible. We therefore also introduce persistent shocks into the basic model. We prove that as the period length shortens, the probability of sale in a given period falls to zero, while the probability of sale per unit of calendar time converges to a well-behaved number.

We start by introducing persistent shocks into the discrete time model. Assume that $s_t \in \{l, h\}$ follows a first order stochastic Markov process; let $\pi_{ss'}$ denote the probability that the state next period is s' given that the current state is s. A partial equilibrium with a fixed value of $\lambda \geq 0$ is still characterized by a set $\mathbb{P} \subset \mathbb{R}_+$, a function $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$, and a function $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, and a pair of value functions $\{v_{s,j}\} \in \mathbb{R}_+^J$ that represent the value of a type s consumer holding a type j tree. In partial equilibrium, buyers and sellers optimize and beliefs are rational. A competitive equilibrium fixes a value of λ and a measure over prices such that markets clear.

To simplify the exposition, we focus on parameters such that in equilibrium $\lambda = 0$, so buyers are indifferent between buying trees and consuming. In this case, problem (P-j) becomes

$$
\begin{aligned}
v_{l,j} = \delta_j + \beta_l(\pi_{ll} v_{l,j} + \pi_{lh} v_{h,j}) &+ \max_{p,\theta} \Big(\min\{\theta, 1\}(p - \beta_l(\pi_{ll} v_{l,j} + \pi_{lh} v_{h,j}))\Big) \\
\text{s.t. } & p \leq \beta_h(\pi_{hl} v_{l,j} + \pi_{hh} v_{h,j}), \\
& v_{l,j'} \geq \delta_{j'} + \beta_l(\pi_{ll} v_{l,j'} + \pi_{lh} v_{h,j'}) + \min\{\theta, 1\}(p - \beta_l(\pi_{ll} v_{l,j'} + \pi_{lh} v_{h,j'})) \text{ for all } j' < j
\end{aligned}
$$

and $v_{h,j} = \delta_j + \beta_h(\pi_{hl} v_{l,j} + \pi_{hh} v_{h,j})$.

As before, the worst type of tree can be sold with probability 1 at a price that leaves buyers' indifferent about purchasing the tree:

$$
p_1 = \frac{\beta_h \delta_1}{1 - \beta_h} \text{ and } v_{l,1} = v_{h,1} = \frac{\delta_1}{1 - \beta_h}.
$$

For higher types of trees, the price and sale probabilities are pinned down by the relevant constraints and Bellman equations:

$$
\begin{gathered}
v_{l,j} = \delta_j + \beta_l(\pi_{ll} v_{l,j} + \pi_{lh} v_{h,j}) + \theta_j(p_j - \beta_l(\pi_{ll} v_{l,j} + \pi_{lh} v_{h,j}))\big) \\
v_{h,j} = \delta_j + \beta_h(\pi_{hl} v_{l,j} + \pi_{hh} v_{h,j}) \\
p_j = \beta_h(\pi_{hl} v_{l,j} + \pi_{hh} v_{h,j}), \\
\Theta(p_j)(p_j - \beta_l(\pi_{ll} v_{l,j-1} + \pi_{lh} v_{h,j-1})) = \Theta(p_{j-1})(p_{j-1} - \beta_l(\pi_{ll} v_{l,j-1} + \pi_{lh} v_{h,j-1}))
\end{gathered}
$$

For $p < p_1$, $\Theta(p) = \infty$ and $\gamma_j(p)$ is defined arbitrarily. To characterize $p > p_1$, focus on the case where the trees are dense. Then the last constraint reduces to

$$\Theta'(p_j)(p_j - \beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j})) + \Theta(p_j) = 0$$

Solve the first three constraints for $v_{l,j}$, $v_{h,j}$, and δ_j to get an expression for $\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j}$. Then solving the differential equation gives

$$\Theta(p) = \frac{1 - \beta_l(1 - \pi_{lh} - \pi_{hl})}{\left(\frac{p}{p_1}\right)^{\frac{\beta_h(1-\beta_l(1-\pi_{lh}-\pi_{hl}))}{\beta_h-\beta_l}} - \beta_l(1 - \pi_{lh} - \pi_{hl})} \in [0, 1]$$

for $p > p_1$. Also solve those same equations for $D(p)$, the type of tree sold at price $p > p_1$:

$$D(p) = p\left(\frac{1 - \big(\beta_l(1 - \pi_{lh}) - \beta_h\pi_{hl}\big)(1 - \Theta(p))}{\beta_h(1 - \beta_l(1 - \Theta(p))(1 - \pi_{lh} - \pi_{hl}))} - 1\right).$$

These are natural generalizations of equations (8) and (9).

We now take the continuous time limit of this model. Define discount rates ρ_s and transition rates q_{hl} and q_{lh} as

$$\rho_s = \frac{1 - \beta_s}{\Delta}, \ q_{hl} = \frac{\pi_{hl}}{\Delta}, \text{ and } q_{lh} = \frac{\pi_{lh}}{\Delta}.$$

We think of $1/\Delta$ as the number of periods within a unit of calendar time. With fixed values of ρ_s, q_{hl}, and q_{lh}, the limit as $\Delta \to 0$ (and so $\beta_s \to 1$ and π_{hl} and $\pi_{lh} \to 0$) then corresponds to the continuous time limit of the model. We find that in this limit, $\Theta(p) \to 0$ but the sale rate per unit of time does not:

$$\lim_{\Delta \to 0} \frac{\Theta(p)}{\Delta} = \frac{q_{hl} + q_{lh} + \rho_l}{\left(\frac{p}{p_1}\right)^{\frac{q_{hl}+q_{lh}+\rho_l}{\rho_l-\rho_h}} - 1} \geq 0$$

for all $p \geq p_1 \equiv \delta_1/(\Delta\rho_s)$, where δ_1/Δ is the minimum dividend per unit of time. From the perspective of a seller, $\frac{\Theta(p)}{\Delta}$ is the arrival rate of a Poisson process that permits her to sell at a price p. Equivalently, the probability that she fails to sell at a price $p > p_1$ during a unit of elapsed time is

$$\exp\left(-\frac{q_{hl} + q_{lh} + \rho_l}{\left(\frac{p}{p_1}\right)^{\frac{q_{hl}+q_{lh}+\rho_l}{\rho_l-\rho_h}} - 1}\right),$$

an increasing function of p that converges to 1 as p converges to infinity and is well-behaved in the limiting economy.

We can also simplify the expression for the type of tree sold at price p:

$$\lim_{\Delta \to 0} \frac{D(p)}{\Delta p} = \rho_h + \frac{q_{hl}(\rho_l - \rho_h)\left(1 - \left(\frac{p}{p_1}\right)^{-\frac{q_{hl}+q_{lh}+\rho_l}{\rho_l - \rho_h}}\right)}{q_{hl} + q_{lh} + \rho_l}.$$

For the lowest type of tree, this confirms that the price-dividend per unit of time ratio is $1/\rho_h$, while it is lower for higher priced trees, reflecting their illiquidity. Only in the special case where $q_{hl} = 0$, so buyers never anticipate needing to sell their trees, is the price-dividend ratio constant at $1/\rho_h$.

The economy in the continuous time limit looks slightly different than the discrete time model. We again imagine many marketplaces, each distinguished by a different price p. Sellers try to sell their trees in the appropriate market, while buyers bring some of their fruit to all of the markets and consume the rest (since we are focusing on the partial equilibrium with $\lambda = 0$). In all but the worst market, there is always too little fruit to purchase all of the trees. As a result, buyers are able to purchase trees immediately, but sellers are rationed and get rid of their trees only at a Poisson rate. Thus it looks as if sellers attempt to sell their stock of trees to the inflow of new fruit from buyers. With insufficient fruit to purchase the trees, the sellers are always rationed. Of course, a seller could immediately sell her trees for the low price p_1, but she chooses not to do so.

The important point is that the frictions generated by adverse selection do not disappear when the period length is short. Intuitively, it must take a real amount of calendar time to sell a tree at a high price or the owners of low quality trees would misrepresent them as being of high quality. This is in contrast to models where trading is slow because of search frictions. In such a framework, the extent of search frictions governs the speed of trading and as the number of trading opportunities per unit of calendar time increases, the relevant frictions naturally disappear.

7 Discussion

Our model shows how prices can be used to separate trees with different qualities. High quality trees trade at a higher price but the market is less liquid. A seller could always choose to sell them at a lower price, but in equilibrium she prefers not to do so.

Our model offers a theory of fire sales following the realization that there may be an adverse selection problem in a market. Imagine that we start from a situation in which everybody thinks that all the trees are valued δ^* so that the price is $p^* = \delta^* \beta_h / (1 - \beta_h)$. Suddenly people learn that there is dispersion in the quality of trees. To be concrete, suppose that the average quality of trees is still δ^*, but all consumers learn that some trees only generate dividend $\delta_1 < \delta^*$, while others generate a higher dividend $\delta_2 > \delta^*$. Information is valuable in this environment, and so assuming it is costless, all consumers would choose to learn whether their trees are high or low quality. But once sellers have learned the quality of their trees, there is an adverse selection problem. Naturally the price of type 1 trees falls; they are known to be of lower quality than before. Interestingly, the price of type 2 trees may fall as well. This happens because in the new equilibrium, type 2 trees only sell probabilistically, reducing their liquidity and driving down the price that buyers are willing to pay for those trees. Thus the revelation of information generates an event that looks like a fire sale. The price of all assets declines and the liquidity of the market starts to dry up. Meanwhile, buyers wait on the sidelines, possibly consuming their fruit, despite the decline in price of all types of trees, good and bad.

Our model also suggests that asset purchase programs, such as the original vision of the Troubled Asset Relief Program in 2008, may alleviate adverse selection problems. Suppose a player with substantial fruit holdings, the "government" to be concrete, announces that it is willing to purchase any asset at a price $\bar{p}$. All sellers with trees that would sell for less that $\bar{p}$, i.e. with $\delta < \bar{\delta} \equiv \bar{p}(1 - \beta_h)/\beta_h$, take them to the government, which then takes a loss on these transactions and perhaps destroys the trees. But the program has the desired effect. It is now common knowledge that the worst type of tree produces dividend $\bar{\delta} > \underline{\delta}$. This tree sells for sure at price $\bar{p}$, while the liquidity and price of all better trees both jump up. Indeed, there are trees that sold for less than $\bar{p}$ before the government announced its program which are not sold to the government but instead become more liquid and experience an increase in price to something in excess of $\bar{p}$. This was exactly the original intent of the TARP; however, the program was never implemented as planned and so we do not know whether it would have successfully moderated adverse selection problems in the market for asset-backed securities.

A Proofs

Proof of Proposition 1. Throughout this proof, let $\bar{\Theta}(p) \equiv \max\{\Theta(p), 1\}$ and $\underline{\Theta}(p) = \min\{\Theta(p), 1\}$. Fix Θ and Γ and take any positive-valued numbers $\{v_j\}$, $\{w_{s,j}\}$, and λ that solve the Bellman equations (4), (5), (6), and (7) for $s = l, h$. Let p_h be an optimal price for buying trees,

$$p_h \in \arg\max_p \left(\bar{\Theta}(p)^{-1} \left(\frac{\beta_h \sum_{j=1}^J \gamma_j(p) v_j}{p} - 1 \right) \right).$$

Similarly let $p_{l,j}$ be an optimal price for selling type j trees,

$$p_{l,j} = \arg\max_p \underline{\Theta}(p)\big(p - \beta_l v_j\big)$$

for all δ. We seek to prove that $V^*(\{k_j\}) \equiv \sum_{j=1}^J v_j k_j$.

If $\lambda = 0$, equations (4), (5), and (6) imply

$$v_j = \pi_h\big(\delta_j + \beta_h v_j\big) + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j} + (1 - \underline{\Theta}(p_{l,j}))\beta_l v_j\big).$$

for all δ. Equivalently,

$$v_j = \frac{\delta_j + \pi_l \underline{\Theta}(p_{l,j})p_{l,j}}{1 - \pi_h\beta_h - \pi_l\beta_l(1 - \underline{\Theta}(p_{l,j}))} > 0.$$

Alternatively, if $\lambda > 0$, the same equations imply

$$v_j = \pi_h\left(\delta_j\left(\left(1 - \bar{\Theta}(p_h)^{-1}\right) + \bar{\Theta}(p_h)^{-1}\frac{\beta_h \sum_{j'=1}^J \gamma_{j'}(p_h)v_{j'}}{p_h}\right) + \beta_h v_j\right) \\ + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j} + (1 - \underline{\Theta}(p_{l,j}))\beta_l v_j\big)$$

for all δ. Equivalently, since $v_j > 0$ by assumption,

$$v_j\left(1 - \pi_h\beta_h - \pi_l\beta_l(1 - \underline{\Theta}(p_{l,j})) - \pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j \sum_{j'=1}^J \gamma_{j'}(p_h)v_{j'}}{p_h v_j}\right) \\ = \pi_h\delta_j\left(1 - \bar{\Theta}(p_h)^{-1}\right) + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j}\big).$$

The right hand side of this expression is positive for all j and so v_j is positive when $\lambda > 0$ if and only if

$$1 - \pi_h\beta_h - \pi_l\beta_l(1 - \underline{\Theta}(p_{l,j})) > \pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j \sum_{j'=1}^J \gamma_{j'}(p_h)v_{j'}}{p_h v_j}. \tag{10}$$

If this restriction fails at any prices p_h and $p_{l,j}$, it is possible for a consumer to obtain unbounded expected utility by buying and selling trees at the appropriate prices. We are interested in cases in which it is satisfied.

Next, let $V(\{k_j\}) = \sum_{j=1}^J v_j k_j$ and $W_s(\{k_j\}) \equiv \sum_{j=1}^J w_{s,j}k_j$ for $s = l, h$. It is straightforward to prove that that V and W_s solve equations (1), (2), and (3) and that the same policy is optimal. (Include proof?)

Finally, we adapt Theorem 4.3 from Werning (2009), which states the following: suppose $V(k)$ for all k satisfies the recursive equations (1), (2), and (3) and there exists a plan that is optimal given this value function which gives rise to a sequence of tree holdings $\{k^*_{i,j,t}(s^{t-1})\}$ satisfying

$$\lim_{t\to\infty}\sum_{s^t}\left(\prod_{\tau=0}^{t-1}\pi_{s_\tau}\beta_{s_\tau}\right)V(\{k^*_{i,j,t}(s^{t-1})\})=0. \tag{11}$$

Then, $V^*=V$.

If $\lambda=0$, an optimal plan is to sell type j trees at price $p_{l,j}$ when impatient and not to purchase trees when patient. This gives rise to a non-increasing sequence for tree holdings. Given the linearity of V, condition (11) holds trivially.

If $\lambda>0$, it is still optimal to sell type j trees at price $p_{l,j}$ when impatient , but patient consumers purchase trees at price p_h and do not consume. Thus

$$\begin{aligned}
k'_{h,j}&=k_j+\bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^J\delta_{j'}k_{j'}}{p_h}\\
k'_{l,j}&=\big(1-\underline{\Theta}(p_{l,j})\big)k_j.
\end{aligned}$$

Using linearity of the value function, the expected discounted value next period of a consumer with tree holdings $\{k_j\}$ this period is

$$\begin{aligned}
&\sum_{j=1}^J v_j\big(\pi_h\beta_hk'_{h,j}+\pi_l\beta_lk'_{l,j}\big)\\
&\quad=\sum_{j=1}^J v_j\left(\pi_h\beta_h\left(k_j+\bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^J\delta_{j'}k_{j'}}{p_h}\right)+\pi_l\beta_l\big(1-\underline{\Theta}(p_{l,j})\big)k_j\right)\\
&\qquad=\sum_{j=1}^J v_jk_j\left(\pi_h\beta_h+\pi_l\beta_l\big(1-\underline{\Theta}(p_{l,j})\big)+\pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j\sum_{j'=1}^J\gamma_{j'}(p_h)v_{j'}}{p_hv_j}\right),
\end{aligned}$$

where the second equality simply rearranges terms in the summation. Equation (11) implies that each term of this sum is strictly smaller than v_jk_j. This implies that there exists an $\eta<1$ such that

$$\eta>\frac{\sum_{j=1}^J v_j\big(\pi_h\beta_hk'_{h,j}+\pi_l\beta_lk_{l,j}\big)}{\sum_{j=1}^J v_jk_j}=\frac{\pi_h\beta_hV(\{k'_{h,j}\})+\pi_l\beta_lV(\{k'_{l,j}\})}{V(\{k_j\})},$$

and so condition (11) holds. ■

Proof of Lemma 1. Write the optimization part of Problem (P-1) as

$$\max_{p,\theta}\big(\min\{\theta,1\}(p-\beta_l v_1)\big) \tag{P-1}$$
$$\text{s.t. } \lambda \le \min\{\theta^{-1},1\}\left(\frac{\beta_h v_1}{p}-1\right).$$

Raising p increases the objective function and tightens the constraint, which ensures the constraint binds. Substituting the binding constraint into the objective function and eliminating the price gives

$$\max_{\theta\ge 0}\min\{\theta,1\}\left(\frac{\beta_h\min\{\theta^{-1},1\}}{\min\{\theta^{-1},1\}+\lambda}-\beta_l\right)v_1$$

If $\lambda=0$, any $\theta_1\ge 1$ attains the maximum. If $\lambda=\beta_h/\beta_l-1$, any $\theta_1\in[0,1]$ attains the maximum. For intermediate values of λ, the unique maximizer is $\theta_1=1$. In any case, the corresponding price and value are

$$p_1=\frac{\delta_1\beta_h(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)} \text{ and } v_1=\frac{\delta_1(1+\lambda)(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)}.$$

We now proceed by induction. Fix $j\ge 2$ and assume for all $j'\in\{2,\dots,j-1\}$, we have established the characterization of $p_{j'}$, $\theta_{j'}$, $v_{j'}$, and $w_{l,j'}$ in the statement of the lemma. We establish the result for j. Setting $(\theta,p)=(\theta_{j-1},p_{j-1})$ is feasible but not generally optimal. Indeed, it delivers a value $v_j>v_{j-1}$ and leaves the constraint $\lambda\le\min\{\theta^{-1},1\}(\beta_h v_j/p-1)$ slack. So consider reducing θ and increasing p while keeping $\min\{\theta,1\}(p-\beta_l v_{j-1})$ constant. This raises the value of the objective function, leaves the constraints for $j'<j-1$ slack, and tightens the constraint $\lambda\le\min\{\theta^{-1},1\}(\beta_h v_j/p-1)$. The optimal policy is therefore achieved when $\lambda=\min\{\theta_j^{-1},1\}(\beta_h v_j/p_j-1)$ and $\min\{\theta_{j-1},1\}(p_{j-1}-\beta_l v_{j-1})=\min\{\theta_j,1\}(p_j-\beta_l v_{j-1})$. Moreover, even if $\theta_{j-1}>1$, it is always the case that $\theta_j<1$ at this solution, allowing a further simplification of these expressions.

Finally, we need to prove that there is a unique value of $v_j>v_{j-1}$ that solves these equations. Eliminate p_j and θ_j from the Bellman equation for v_j with the binding constraints:

$$(1-\pi_h\beta_h-\pi_l\beta_l)v_j=\delta_j(1+\pi_h\lambda)+\pi_l\min\{\theta_{j-1},1\}\frac{\big(\beta_h-\beta_l(1+\lambda)\big)^2 v_{j-1}v_j}{\big(\beta_h v_j-\beta_l(1+\lambda)v_{j-1}\big)(1+\lambda)}. \tag{12}$$

If $\lambda=\beta_h/\beta_l-1$, the last term is zero and so this pins down v_j uniquely. Otherwise we prove that there is a unique solution to equation (12) with $v_j>v_{j-1}$. In particular, the left hand side is a linearly increasing function of v_j, while the right hand side is an increasing, concave function, and so there are at most two solutions to the equation. As $v_j \to \infty$, the left hand side exceeds the right hand side, and so we simply need to prove that as $v_j \to v_{j-1}$, the right hand side exceeds the left hand side.

First assume $j = 2$ so $\theta_{j-1} = \theta_1 \geq 1$. Then we seek to prove that

$$(1 - \pi_h\beta_h - \pi_l\beta_l)v_1 < \delta_2(1 + \pi_h\lambda) + \pi_l \frac{\big(\beta_h - \beta_l(1+\lambda)\big)v_1}{1+\lambda}.$$

Since $v_1 = \frac{\delta_1(1+\lambda)(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)}$ and $\delta_1 < \delta_2$, we can confirm this directly. Next take $j \geq 3$. In this case, in the limit with $v_j \to v_{j-1}$, the right hand side of (12) converges to

$$\begin{aligned}\delta_j(1 + \pi_h\lambda) &+ \pi_l \min\{\theta_{j-1}, 1\} \frac{\big(\beta_h - \beta_l(1+\lambda)\big)v_{j-1}}{(1+\lambda)} \\ &> \delta_{j-1}(1 + \pi_h\lambda) + \pi_l \min\{\theta_{j-2}, 1\} \frac{\big(\beta_h - \beta_l(1+\lambda)\big)^2 v_{j-2}v_{j-1}}{\big(\beta_h v_{j-1} - \beta_l(1+\lambda)v_{j-2}\big)(1+\lambda)},\end{aligned}$$

where the inequality uses the indifference condition

$$\min\{\theta_{j-2}, 1\}(p_{j-2} - \beta_l v_{j-2}) = \min\{\theta_{j-1}, 1\}(p_{j-1} - \beta_l v_{j-2})$$

and the assumption $\delta_{j-1} < \delta_j$. Rewriting equation (12) for type $j-1$,

$$(1 - \pi_h\beta_h - \pi_l\beta_l)v_{j-1} = \delta_{j-1}(1 + \pi_h\lambda) + \pi_l \min\{\theta_{j-2}, 1\} \frac{\big(\beta_h - \beta_l(1+\lambda)\big)^2 v_{j-2}v_{j-1}}{\big(\beta_h v_{j-1} - \beta_l(1+\lambda)v_{j-2}\big)(1+\lambda)},$$

it follows that

$$(1 - \pi_h\beta_h - \pi_l\beta_l)v_{j-1} < \delta_j(1 + \pi_h\lambda) + \pi_l \min\{\theta_{j-1}, 1\} \frac{\big(\beta_h - \beta_l(1+\lambda)\big)v_{j-1}}{(1+\lambda)},$$

which completes the step.

Finally, set $p_j = \beta_h v_j/(1+\lambda) > p_{j-1}$ and $\theta_j = \min\{\theta_{j-1}, 1\}(p_{j-1} - \beta_l v_{j-1})/(p_j - \beta_l v_{j-1}) < \theta_{j-1}$, completing the proof. ■

Proof of Proposition 2.

We first prove that the solution to Problems (P-j) describe a partial equilibrium and then prove that there is no other equilibrium.

Existence. As described in the statement of the proposition, we look for a partial equilibrium where $\mathbb{P} = \{p_j\}$, $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, and v_j and $w_{l,j}$ solve Problem (P-j). Also for notational convenience define $p_{J+1} = \infty$. To complete the characterization, we define Θ and Γ on their full support $\mathbb{R}_+$. For $p < p_1$, $\Theta(p) = \infty$ and $\Gamma(p)$ can be chosen arbitrarily, for example $\gamma_1(p) = 1$. For $j \in \{1, \ldots, J\}$ and $p \in (p_j, p_{j+1})$, $\Theta(p)$ satisfies sellers' indifference condition $w_{l,j} = \delta_j + \beta_l v_j + \Theta(p)(p - \beta_l v_j)$ and $\gamma_j(p) = 1$. To prove that this is an equilibrium, we simply verify that under these beliefs, the three conditions in the definition of partial equilibrium hold. The first condition, consistency of the value functions, holds by construction.

The second condition is sellers' optimality. By construction, for all $j \in \{1, \ldots, J\}$ and $p \geq p_1$, $w_{l,j} = \delta_j + \beta_l v_j + \Theta(p)(p - \beta_l v_j)$ if $\gamma_j(p) = 1$. We must only show that $w_{l,j} \geq \delta_j + \beta_l v_j + \Theta(p)(p - \beta_l v_j)$ for all other p.

To prove this, first take any $j \in \{2, \ldots, J\}$, $j' < j$, and $p \in (p_j, p_{j+1})$. By construction, $\Theta(p_j)(p_j - \beta_l v_j) = \Theta(p)(p - \beta_l v_j)$. Since v is increasing in j and $p > p_j$, it follows that $\Theta(p_j)(p_j - \beta_l v_{j'}) > \Theta(p)(p - \beta_l v_{j'})$. Also by the construction in Problem (P-j), $\Theta(p_{j'})(p_{j'} - \beta_l v_{j'}) \geq \Theta(p_j)(p_j - \beta_l v_{j'})$. Combining inequalities gives $\Theta(p_{j'})(p_{j'} - \beta_l v_{j'}) > \Theta(p)(p - \beta_l v_{j'})$ for all $p \in (p_j, p_{j+1})$ and $j' < j$.

Similarly, take any $j \in \{1, \ldots, J-1\}$, $j' > j$, and $p \in (p_j, p_{j+1})$. By construction, $\Theta(p_{j+1})(p_{j+1} - \beta_l v_j) = \Theta(p)(p - \beta_l v_j)$, since type j is indifferent about the price p_{j+1}. Since v is increasing in j and $p < p_{j+1}$, it follows that $\Theta(p_{j+1})(p_{j+1} - \beta_l v_{j'}) > \Theta(p)(p - \beta_l v_{j'})$. Also by the construction in Problem (P-$(j+1)$), $\Theta(p_{j'})(p_{j'} - \beta_l v_{j'}) \geq \Theta(p_{j+1})(p_{j+1} - \beta_l v_{j'})$. Combining inequalities gives $\Theta(p_{j'})(p_{j'} - \beta_l v_{j'}) > \Theta(p)(p - \beta_l v_{j'})$ for all $p \in (p_j, p_{j+1})$ and $j' > j$.

Finally we turn to buyers' optimality condition. By construction, the inequality binds at all $p \in \mathbb{P}$. For $p < p_1$, it is satisfied because $\Theta(p)^{-1} = 0$. If $\lambda = 0$, the inequality holds for all $p \in (p_j, p_{j-1})$ because $\beta_h v_j / p_j = 1$ and so $\beta_h v_j / p < 1$. If $\lambda > 0$, Lemma 1 implies $\min\{\Theta(p)^{-1}, 1\} = 1$ for all $p \geq p_1$ and $\beta_h v_j / p_j < \beta_h v_j / p$ for all $p \in (p_j, p_{j-1})$, and so the inequality again holds for all $p > p_1$.

Uniqueness. Now take any partial equilibrium $\{v, w_l, \mu . \Theta, \Gamma\}$. We first claim that v is increasing in j. This follows immediately from part 1 of the definition of equilibrium: Let $p_{j'}$ denote the price offered by j'. For $j > j'$, it is feasible to offer the same price $p_{j'}$, and since $\delta_j > \delta_{j'}$, this gives a higher value $v_j > v_{j'}$. Behaving optimally gives a still higher value.

We next claim that for each $j \in \{1, \ldots, J\}$, there exists a price $p_j \in \mathbb{P}$ with $\theta_j \equiv \Theta(p_j) < \infty$ and $\gamma_j(p_j) > 0$. (Proof?)

In the remainder of the proof, we take any $j \in \{1, \ldots, J\}$ and $p_j \in \mathbb{P}$ with $\theta_j = \Theta(p_j) < \infty$

and $\gamma_j(p_j) > 0$. First we prove that the constraint $\lambda \le \min\{\theta_j^{-1}, 1\}(\beta_h v_j/p_j - 1)$ is satisfied. Second we prove that the constraint $w_{l,j'} \ge \delta_{j'} + \beta_l v_{j'} + \min\{\theta_j, 1\}(p_j - \beta_l v_{j'})$ is satisfied for all $j' < j$. Third we prove that the pair (θ_j, p_j) delivers value $w_{l,j}$ to sellers of type j trees. Fourth we prove that (θ_j, p_j) solves (P-j).

Step 1. We prove that $\lambda \le \min\{\theta_j^{-1}, 1\}(\beta_h v_j/p_j - 1)$. To derive a contradiction, assume $\lambda > \min\{\theta_j^{-1}, 1\}(\beta_h v_j/p_j - 1)$. Equilibrium condition (iii) implies that there is a j' with $\gamma_{j'}(p_j) > 0$ and $\lambda < \min\{\theta_j^{-1}, 1\}(\beta_h v_{j'}/p_j - 1)$. Then for all $p' > p_j$ and $\theta' = \Theta(p')$,

$$\min\{\theta', 1\}(p' - \beta_l v_{j'}) \le \min\{\theta_j, 1\}(p_j - \beta_l v_{j'}) < \min\{\theta_j, 1\}(p' - \beta_l v_{j'}).$$

The weak inequality holds from type j' sellers' optimality condition, since p_j is an optimal price for type j' sellers, while the strict inequality uses $p' > p_j$. This implies $\theta' < \theta_j$. Next observe that for all $j'' < j'$,

$$\min\{\theta', 1\}(p' - \beta_l v_{j''}) < \min\{\theta_j, 1\}(p_j - \beta_l v_{j''}) \le w_{j''} - \delta_{j''} - \beta_l v_{j''},$$

where the first inequality follows because $p' > p_j$ and $v_{j''} < v_{j'}$ and the second follows from type j'' sellers' optimality condition. This implies that $\gamma_{j''}(p') = 0$. Thus any $p' > p_j$ only attracts type j' sellers or higher and so delivers value at least equal to $\min\{\theta'^{-1}, 1\}(\beta_h v_{j'}/p' - 1)$ to buyers. For p' sufficiently close to p_j, this exceeds λ, contradicting buyers' optimality.

Step 2. Again take any $j \in \{1, \ldots, J\}$ and $p_j \in \mathbb{P}$ with $\theta_j = \Theta(p_j) < \infty$ and $\gamma_j(p_j) > 0$. Sellers' optimality implies $w_{l,j'} \ge \delta_{j'} + \beta_l v_{j'} + \min\{\theta_j, 1\}(p_j - \beta v_{j'})$ for all j', and so the second constraint is satisfied.

Step 3. Again take any $j \in \{1, \ldots, J\}$ and $p_j \in \mathbb{P}$ with $\theta_j = \Theta(p_j) < \infty$ and $\gamma_j(p_j) > 0$. Sellers' optimality implies $w_{l,j} = \delta_j + \beta_l v_j + \min\{\theta_j, 1\}(p_j - \beta v_j)$ for all j, and so the policy delivers value $w_{l,j}$.

Step 4. Suppose there is a policy (θ, p) that satisfies the constraints of problem (P-j) and delivers a higher payoff. That is,

$$w_{l,j} < \delta_j + \beta_l v_j + \min\{\theta, 1\}(p - \beta_l v_j)$$
$$\lambda \le \min\{\theta^{-1}, 1\}(\beta_h v_j/p - 1)$$
$$w_{l,j'} \ge \delta_{j'} + \beta_l v_{j'} + \min\{\theta, 1\}(p - \beta_l v_{j'}) \text{ for all } j' < j.$$

Choose $p' < p$ such that

$$w_{l,j} < \delta_j + \beta_l v_j + \min\{\theta, 1\}(p' - \beta_l v_j)$$
$$\lambda < \min\{\theta^{-1}, 1\}(\beta_h v_j / p' - 1)$$
$$w_{l,j'} > \delta_{j'} + \beta_l v_{j'} + \min\{\theta, 1\}(p' - \beta_l v_{j'}) \text{ for all } j' < j.$$

Now sellers' optimality condition implies

$$w_{l,j} \geq \delta_j + \beta_l v_j + \min\{\Theta(p'), 1\}(p' - \beta_l v_j),$$

so $\Theta(p') < \theta$. This implies that

$$w_{l,j'} > \delta_{j'} + \beta_l v_{j'} + \min\{\Theta(p'), 1\}(p' - \beta_l v_{j'}) \text{ for all } j' < j,$$

and so in particular $\gamma_{j'}(p') = 0$ for all $j' < j$. But then

$$\lambda < \min\{\Theta(p')^{-1}, 1\} \left(\frac{\beta_h v_j}{p'} - 1 \right) \leq \min\{\Theta(p')^{-1}, 1\} \left(\frac{\beta_h \sum_{j'=1}^{J} \gamma_{j'}(p') v_{j'}}{p'} - 1 \right),$$

a violation of buyers' optimality condition. This completes the proof. ■

Proof of Proposition 3. We prove that there exists a unique $\lambda \in [0, \beta_h/\beta_l - 1]$ such that the market clearing condition holds. ■

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