CENTRAL BANK OF CYPRUS
EUROSYSTEM

WORKING PAPER SERIES

On Imperfect Competition with Occasionally Binding Cash-in-Advance Constraints

Huw Dixon

Panayiotis M. Pourpourides

August 2011

Working Paper 2011-3

Address
80 Kennedy Avenue
CY-1076 Nicosia, Cyprus

Postal Address
P. O. Box 25529
CY-1395 Nicosia, Cyprus

E-mail
publications@centralbank.gov.cy

Website
http://www.centralbank.gov.cy

Fax
+357 22 378153

Papers in the Working Paper Series may be downloaded from:
http://www.centralbank.gov.cy/nqcontent.cfm?a_id=5755

On Imperfect Competition with Occasionally Binding Cash-in-Advance Constraints

Huw Dixon* **and Panayiotis M. Pourpourides****

August 2011

Abstract

We provide extensive theoretical analysis of the general equilibrium of an economy with imperfect competition in the final goods sector, endogenous production and fully flexible prices in the presence of occasionally binding cash-in-advance (CIA) constraints, under general assumptions about the velocity of money. Whether the CIA constraint binds or not and the induced variability of the velocity of money depend on expectations of risk-averse consumers about the future relative value of money as well as the degree of imperfect competition. We establish the uniqueness of the equilibrium, the conditions under which money has real effects, even in the absence of other real assets, and examine the role of imperfect competition and welfare implications. With perfect foresight, in a zero-inflation steady state the CIA constraint strictly binds and output is less than would occur when the CIA constraint is non-binding. There is also an optimal negative steady-state inflation rate. Finally, we consider how the introduction of capital and bonds would fit into the framework.

Keywords: Cash-in-advance, general equilibrium, monopolistic competition, imperfect competition, money velocity.

JEL Classification: D43, E31, E41, E51.

---

* Cardiff Business School, Cardiff University, ** Cardiff Business School, Cardiff University and Central Bank of Cyprus. The opinions in this paper are solely those of the authors and should not be interpreted as reflecting the views of the Central Bank of Cyprus or the Eurosystem.

Correspondence: Huw Dixon, Cardiff Business School, Cardiff University, Aberconway Building, Colum Drive, Cardiff, CF10 3EU, UK. E-mail: DixonH@cardiff.ac.uk, Panayiotis M. Pourpourides, Economic Research Department, Central Bank of Cyprus, 80 Kennedy Avenue, P.O.Box 25529, 1395 Nicosia, Cyprus, E-mail: PanayiotisPourpourides@centralbank.gov.cy or Cardiff Business School, Cardiff University, Aberconway Building, Colum Drive, Cardiff, CF10 3EU, UK. E-mail: pourpouridesp@cardiff.ac.uk

1 Introduction

We introduce and explore an analytically tractable dynamic general equilibrium model of monopolistic competition with endogenous production and a money demand that arises from an occasionally binding cash-in-advance (CIA) constraint.[1] Monopolistic competition arises in the final goods sector which comprises of a finite number of firms. Whether the CIA constraint binds is endogenous and depends on expectations of risk-averse consumers about the future relative value of money as well as the degree of imperfect competition. We show that the velocity of money has a specific upper bound that is reached whenever the CIA constraint binds. We demonstrate that money can have real effects without requiring the presence of other physical assets or restrictions on how assets are used for transactions: even when money is the only asset in the economy, monetary policy is non-trivial.[2]

Although the nominal wages and prices are fully flexible, we show that there are cases where prices exhibit a sluggish response to a change in money supply. As is well known, the CIA constraint creates a transactions demand for money even though money provides no utility.[3] To see why the CIA constraint might not bind, note that with uncertainty about

[1] Cash-in-advance models continue to be widely used in monetary economics, e.g. Schmitt-Grohé and Uribe (2000), Dotsey and Sarte (2000), Adão, Correia and Teles (2003), Ireland (2003, 2005), Bloise and Polemarchakis (2006), Santos (2006), Evans, Honkapohja, Marimon (2007), Devereux and Siu (2007), Díaz-Giméneza, Giovannettic, Marimon and Teles (2008), Hromcová (2008), Chen and Li (2008), Alvarez, Atkeson and Edmond (2009), Giraud and Tsomocos (2010), Adão, Correia and Teles (2011). Some of these papers mainly focus on the case of a binding CIA constraint and constant velocity of money, e.g. Dotsey and Sarte (2000), Santos (2006), Evans, Honkapohja, Marimon (2007), Chen and Li (2008), Díaz-Giméneza, Giovannettic, Marimon and Teles (2008).

[2] Chamley and Polemarchakis (1984) note that a standard argument for money non-neutrality in a general equilibrium framework lies on the existence of other real assets. Changes in the money supply affect the price level which in turn affects the return of money as an asset relative to the other physical assets. As a result, individuals realign their portfolios and the equilibrium holdings of physical assets change. Within this framework general equilibrium models require heterogeneous beliefs or other frictions.

[3] This was the rationale behind the first general formulation of the CIA constraint in Grandmont and Younes (1972).

the next period, households may choose to hold money at the end of this period to relax the next periods CIA constraint. In this sense, in a dynamic model the CIA can give rise to a precautionary or buffer-stock motive for holding money over and above the need to finance the current period's transactions. This rationale for holding money is inherently dynamic in nature: money is demanded over and above what is required for financing current transactions not because it provides a flow of current utility, but because it increases expected utility in subsequent periods.[4] The contribution of this paper is to provide a tractable analytical framework for characterizing the solution (in terms of real and nominal variables) both when the CIA constraint binds and when it does not bind and to characterize the conditions determining which case holds. Our results provide a theoretical perspective on a widely used model in applied macroeconomics.

Alvarez, Atkeson and Edmond (2009), consider a CIA economy where production is exogenous and output is modelled as a stochastic endowment process. Their assumption that households are restricted from using funds from interest-bearing accounts for consumption purposes in every period prevents the CIA constraint from binding at all times thus allowing the velocity of money to vary.[5] A direct implication of this is that prices respond sluggishly to changes in money supply because aggregate velocity decreases after an injection of money. They motivate this feature by presenting correlations between velocity and measures of money that exhibit a negative relationship. Chiu (2007), on the other hand, provides ev-

[4] In addition, money might be carried over as a store of value (even in the absence of uncertainty) when.the nominal price of consumption is falling and money has a real rate of return above zero.

[5] In the special case where households can use their brokerage account for consumption in every period, the model reduces to a standard CIA model of the type they consider where the CIA constraint binds at all times and velocity is constant.

idence that cross-country correlations between money and velocity for the OECD countries are all significantly positive. We argue that by merely looking at aggregate correlations in the data, one cannot safely draw conclusions about the direction of the effect of money growth on velocity because velocity is driven by other factors as well. It is possible that money velocity exhibits an overall negative relation with money growth despite the fact that an increase in money supply on its own has a positive effect on velocity. In our model, we allow velocity to be constant, increasing or decreasing to a set of arguments which consists of a number of factors such as velocity-specific shocks, money transfers and technology shocks. In addition, we do not impose any requirement for smoothness or differentiability.

In an earlier paper, Cooley and Hansen (1989), introduce a CIA constraint in a stochastic optimal growth model with endogenous indivisible labor and capital, and perfectly competitive markets[6]. Assuming that the CIA constraint always binds, they find that the impact of money on real quantities is small at business cycle frequencies.[7] They show that the impact of money is significantly large only in the long-run. Another strand of the literature focuses on nominal rigidities of one kind or another which result in real effects of monetary policy in the short-run.[8]

[6]Svensson (1985), intoduced money via a CIA constraint in a general equilibrium model where other financial assets are also traded. Due to the absence of physical capital, the equilibrium consumption always equals output which is specified as a stochastic endowment process. In such setting, it is unclear whether output is dependent or independent of monetary expansion. His model is differentiated from that of Lucas (1982) in that consumers decide on their cash balances before they know the current state of nature and hence before they know their consumption. This feature leads to potential variation in the velocity of money as the CIA constraint is sometimes non-binding.

[7]The impact of money on real variables results from the inflation tax. That is, increases in the growth rate of money lead agents to expect higher inflation. The positive inflation tax on the consumption good induces agents to lower work effort and as a result, output and consumption. In other words, the agents substitute away from activities that involve the use of cash (consumption good) in favor of activities that do not require cash (leisure). Among others, Cooley and Hansen (1995, 1997) adopt a similar framework.

[8]This is the case in the neoclassical synthesis framework (e.g. Don Patinkin 1956) and also the *new* neoclassical synthesis (e.g. Woodford 2003).

In this paper we offer an alternative general equilibrium framework of a CIA economy where output is produced by monopolistic firms. Specifically, risk averse workers supply labor and monopolists set prices and produce output (the labor market is competitive). To keep our analysis simple and tractable and since our objective is to examine the qualitative aspects of money rather than to match features of the data, we abstract from the presence of physical assets such as capital. Focusing on an economy with primitive financial structure also enables us to demonstrate the direct effects of money, rather than those arising from portfolio choice.[9] In section 4, we provide a discussion about how the introduction of real assets such as capital and bonds might influence the results.

We allow for a very general set of possibilities about how the velocity of money is determined. There are three potential sources of uncertainty: random technology innovations, random money transfers and velocity shocks. Money transfers take place at the beginning of the period whereas technology innovations are revealed at the end of the period. The state vector consists of a technology innovation, money balances and a velocity specific-shock. In Proposition 1 we show that velocity has an upper bound (which is decreasing with the elasticity of demand of the consumption good), and in Proposition 2 we characterise the economy when the CIA constraint is binding and when it is not. Cooley and Hansen (1989) suggested in their conclusion (p. 746), "... the most important influence of money on short-run fluctuations are likely to stem from the influence of the money supply process on expectations of relative prices". Here, we establish this argument analytically. When

[9] The assumption that money is the only asset in the economy is not an unusual one in the literature: e.g. Lagos and Wright (2003), Lagos and Rocheteau (2005), Santos (2006).

the CIA constraint is non-binding, the economy is at its efficient output[10] with the Classical feature that money is neutral.[11] This happens when the expected value of money equals its current value (i.e the expected discounted relative price of consumption remains unchanged), so that consumers are indifferent between spending a unit of money today and holding it for one period. However, when particular state vectors occur, the CIA constraint binds because the agents expect that the relative value of money will decrease next period (i.e the expected discounted relative price of consumption will rise). As a result, they rush to spend all their money holdings the current period which leads to an increase in the velocity of money to the extent that it hits its upper bound. In this case, there is a unique equilibrium where money induces real effects: equilibrium output, consumption, work effort and real profits are functions of money balances as well as expectations for the future absolute value of money. The transmission mechanism for money to have real effects is the presence of the CIA constraint, through which the level of the price has a direct effect on consumer demand. This can be viewed as a type of Keynesian *effective demand* mechanism. Furthermore, we show that (for given technology) the level of output, hours worked and consumption is less when the CIA binds than when it is not leading to lower utility (Proposition 3(i)). This inefficiently low level of output occurs because the binding CIA constraint distorts the intra-temporal work-leisure decision and discourages work. Furthermore, there is a precise sense in which the current utility is lower the more the CIA binds (Proposition 3(ii)).

The problem of the monetary authority is not modeled explicitly and money transfers

[10] Note that under the assumptions we make, imperfect competition per se does not alter total output or the labour supply. Rather, it alters the distribution of income between wages and profit.

[11] In other words, real variables are driven only by current technology innovations, whereas money transfers and velocity-specific shocks only affect the price level.

are treated as random variables (with a known distribution) by firm owners and consumers. For illustrative purposes we assume that the velocity of circulation is an increasing function of technology and money transfers.[12] Then, an increase of money supply increases the probability of a binding CIA constraint. We argue that the monetary authority would not necessarily avoid expansionary money supply because, as we show, there are cases where it might be welfare improving. When the monetary authority decides the transfer of money, neither the technology innovation nor the velocity-specific shock are known. Therefore, the transfer may be optimal ex-ante based on current information and expectations but not optimal ex post, after technology and velocity shocks are revealed.

In section 3 we illustrate the scope of the model by looking at the case of perfect foresight. Perfect foresight removes the precautionary/buffer-stock demand for money, but there is still a potential role for money over and above the current transactions demand. In particular we are able to provide conditions relating to whether the current CIA constraint binds or not in terms of the current growth in the money supply (Proposition 7)[13] or inflation and productivity growth (Proposition 9). We show that in a zero-inflation steady state (all real and nominal variables are constant), the CIA constraint always binds. In section 3.1 we consider the general case allowing for non-zero inflation steady-states (all real variables constant, with money and prices growing at a constant rate). In Proposition 10, we show that if inflation is greater than the discount rate minus 1 (a small negative number), the CIA constraint always binds. When the inflation rate equals the discount rate minus 1, the

[12] This assumption is supported by evidence provided in Chiu (2007) and Hromcová (2008).

[13] Proposition 8 gives the corresponding real variables in the case of a binding and non-binding CIA constraint.

CIA constraint never binds.[14] Since utility is higher in the steady-state with the non-binding CIA constraint, it follows that the optimum inflation rate here is negative. The idea is that negative inflation provides a real return to nominal money that exactly offsets the effect of discounting. This has obvious similarities to the Friedman (1969) argument for a negative inflation rate made in the context of a money in the utility function approach.

The rest of the paper is organized as follows. In section 2, we describe the economic environment which includes the problem of the firms, the problem of the workers and the analysis and discussion of the equilibrium conditions. In section 3 we look at the special case of perfect foresight and section 4 briefly examines how the introduction of capital and bonds might influence our results. Section 5 concludes.

2 Model Economy

The economy is populated by risk averse workers and monopolistic firms which are owned by risk-neutral entrepreneurs. There are incomplete financial markets which mean that there is no source of insurance for workers. There is a perfectly competitive labor market and a goods market where the workers and the firms trade labor services and the final good. The agents exchange goods and labor services using cash which is the only medium of exchange. As the quantity theory of money indicates, at the aggregate level, nominal output varies with the nominal money balances times its velocity which can be written as a definition of

[14] No steady-state exists when infation is lower than the discount rate minus 1 (Proposition 10 (*iii*)).

the velocity of money q_t:

$$q_t \equiv \frac{P_t y_t}{\overline{M}_t} \tag{1}$$

where P_t is the aggregate price level, y_t is aggregate real output, $\overline{M}_t$ is the total quantity of money. The velocity q_t is not a choice variable of a single agent but it is rather determined at the aggregate level.

2.1 Firms

There is a number $n > 1$ of firms, each producing a good $x_i \geq 0$, $i = 1, ..., n$. The firms are managed by entrepreneurs that consume the firms' profits. The price for good i is denoted by $p(x_i)$, where $p'(\cdot) \leq 0$ and $p''(\cdot) \leq 0$, while the n-vector of prices is denoted by $\mathbf{p} \in \Re^n_+$ and general price level P: $\Re^n_+ \to \Re_+$. The firm produces output by employing a fixed number $m \geq 1$ of workers, each providing h_i hours of work, via technology $x_i(h_i; m, \theta_i)$, where $x_i'(\cdot) > 0$ and $x_i''(\cdot) \leq 0$. $\theta_i > 0$ is an exogenous productivity shock. The latter is distributed according to the conditional p.d.f. $\vartheta(\widetilde{\theta}; \theta')$ for $\widetilde{\theta} \in \Theta \subset \Re_+$ where θ' denotes the previous period realized value of θ. The objective function of firm i can be written as

$$\Pi_i = p\left(x_i\right) x_i - Pwmh_i \tag{2}$$

where Π_i are profits and w is the real hourly wage rate. The problem of the firm is to maximize its profits by choosing hours, taking as given the aggregate price level and the real hourly wage rate. The necessary and sufficient condition for profit maximization is

$$x_i' \left[p' \left(x_i \right) x_i + p \left(x_i \right) \right] = Pwm \tag{3}$$

In our analysis, we assume that firms employ the same technology and thereby, $\theta_i = \theta$. To obtain an analytical solution we assume that the technology is described by the linear production function $x_i \left(h_i; m, \theta \right) = \theta m h_i$. The inverse demand for good i is also linear and of the form $p_i = A - Bx_i$ where $A > 0$ and $B > 0$. The demand function is a special case generated from a class of linear- homothetic (LH) preferences. LH preferences have the property that the firm's demand curve is linear in its own price treating the general price indexes as given (as in monopolistic competition).[15] The aggregate price level is defined as

$$P \left(\mathbf{p} \right) = \mu + \gamma \left(\mu - s \right) \tag{4}$$

$$\mu = \frac{\sum_{i=1}^{n} p_i}{n} \ , \ s = \left(\frac{\sum_{i=1}^{n} p_i^2}{n} \right)^{\frac{1}{2}}$$

where $\gamma > 1$ (γ is the absolute value of the elasticity of demand when prices are equal). Notice that (4) implies that when all prices are equal then, $p_i = s = \mu = P$. Coefficients A and B correspond to

$$A = \frac{1+\gamma}{\gamma} s \ \ , \ \ B = \frac{snP}{\gamma Y}$$

where Y denotes the economy's total nominal expenditure. Hence we can solve (3) for labor

[15] See Datta and Dixon (2000, 2001).

demand function, nominal price and profits:

$$h = \frac{1}{2Bm\theta}\left[A - P\frac{w}{\theta}\right] \tag{5}$$

$$p_i = \frac{1}{2}\left[A + \frac{w}{\theta}P\right] \tag{6}$$

$$\Pi_i = \frac{1}{4B}\left[A - P\frac{w}{\theta}\right]^2 \tag{7}$$

Since firms face the same technology shock θ, the equilibrium will be symmetric. In other words, in equilibrium all firms will set their price equal to P. Thus, the individual product demand reduces to an expression of aggregate real output, $y = nx$, where $y = Y/P$. Then, condition (6) reduces to

$$w = \frac{\gamma - 1}{\gamma}\theta \tag{8}$$

(8) is the simple condition that the real wage is the marginal product of labor times the inverse of the markup. Condition (5) reduces to

$$h^d = \frac{y}{nm\theta} \tag{9}$$

where h^d denotes labor demand. Finally, real profits per firm reduce to

$$\pi_i = \frac{y}{\gamma n} \tag{10}$$

where $\pi_i = \Pi_i/P$. Aggregate profits are then $\overline{\pi} = n\pi_i$ with the share of total profits in output being $\gamma^{-1} < 1$. Since all profits are consumed by entrepreneurs, it follows that total consumption by worker-households is equal to $(1-\gamma^{-1})\,y$.[16] That is, the share of consumption by worker-households is determined by the elasticity of demand, with a higher consumption share with a higher elasticity. Note that whilst we are interested in the effects of monopolistic competition in our model, it is in no way essential for the non-neutrality of money in this model: non-neutrality of money due to binding CIA constraints is if anything more likely to occur if there is perfect competition and all income takes the form of wages.

2.2 Worker-Consumers

Time is discrete and infinite, $t \in \mathbb{Z}_+ = \{1, 2...\infty\}$. There are ($n$ x m) identical worker-consumers with preferences over leisure, l, and consumption, c. The utility function is given by $u(c_t, l_t) = \ln c_t + \phi \ln l_t$ where $\phi > 0$. Each worker-consumer is endowed with one unit of time which is split between work and leisure that is, $l+h = 1$. Since all worker-households are identical and face the same prices, we shall model them as a representative worker-household (thus avoiding the need for a household subscript and aggregation).

The consumer's wealth constraint is given by

$$M^c_{t+1} + P_t c_t = M^c_t + \nu_t + P_t w_t h_t \tag{11}$$

where $M^c \in \Re_+$ are the consumer's nominal money holdings and ν is a money increase or decrease such that $M^c > |\nu|$. The transfer ν_t is made at the end of period $t-1$ and before θ_t is realized. It takes a while for the transfer to be completed but the timing is such that the

[16] Note that the market becomes more competitve as γ inceases and/or as n increases.

money is available at the beginning of the period. Consumers treat ν as a random variable that is distributed according to $\xi(\widetilde{\nu};\nu')$ for $\widetilde{\nu} \in N$ where ν' denotes the previous period transfer and $N = \{\widetilde{\nu} \in \Re : \nu + M^c > 0\}$. The consumer receives her labor earnings at the end of the period but purchases consumption at the beginning of the period. As a result, she faces a cash-in-advance constraint:

$$P_t c_t \leq M_t^c + \nu_t \tag{12}$$

The problem of the consumer is to choose consumption, labor supply and money balances to maximize utility subject to the budget constraint and the CIA constraint. We will say that the CIA is binding whenever $P_t c_t = M_t^c + \nu_t$. It is *weakly binding* when the household does not wish to consume more; it is *strictly binding* when the household is constrained to consume less than it would like to in the absence of the CIA. As in Svensson (1985), money holdings cannot be adjusted after the state of the economy is known. Unlike Svensson however, the exogenous current transfer of money in our model can be used to buy current consumption. In other words, we do not assume that only money carried over from the previous period is required to finance current consumption.[17]

The Bellman equation associated with the consumer's problem is the following:

$$V\left(M_t^c\right) = \max\{u\left(c_t, l_t\right) + \beta E_t V\left(M_{t+1}^c\right)$$
$$-\lambda_{1t}\left[M_{t+1}^c + P_t c_t - M_t^c - \nu_t - P_t w_t h_t\right] - \lambda_{2t}\left[P_t c_t - M_t^c - \nu_t\right]\}$$

[17] See Walsh (2003, chapter 3.3) for a discussion of alternative assumptions under the CIA constraint.

where β is the discount factor, λ_{1t} is the shadow price of the standard budget constraint and λ_{2t} is the shadow price of the CIA constraint.

This yields the following necessary and sufficient first-order conditions:

$$u_c(c_t, l_t) = \lambda_{1t} P_t + \lambda_{2t} P_t \tag{13}$$

$$u_l(c_t, l_t) = \lambda_{1t} P_t w_t \tag{14}$$

$$\lambda_{1t} = \beta E_t \{\lambda_{1t+1} + \lambda_{2t+1}\} \tag{15}$$

Notice that in equilibrium, $M_t^c = M_t$. Combining (13), (14) and (15) yields

$$\frac{u_l(c_t, l_t)}{w_t} = \beta E_t \left\{ \frac{u_c(c_{t+1}, l_{t+1})}{1 + g_{pt+1}} \right\}$$

where $g_{pt} = P_t / P_{t-1} - 1$ denotes the inflation rate in period t. If the CIA constraint does not bind or is only weakly binding in period t ($\lambda_{2t} = 0$), the left-hand side of the above condition is also equal to the marginal utility of consumption, which implies that the marginal benefit of work will equal the marginal cost of work, i.e $u_c(c_t, l_t)\, w_t = u_l(c_t, l_t)$. On the other hand, if the CIA constraint is strictly binding ($\lambda_{2t} > 0$) then the marginal benefit of work will be greater than the marginal cost of work, i.e $u_c(c_t, l_t)\, w_t > u_l(c_t, l_t)$. Using the fact that utility is separable in consumption and leisure, it is straightforward to show that money demand is governed by[18]

$$E_t\left[\left(\frac{\beta u_c\left(c_{t+1}\right)}{u_c\left(c_t\right)}\right)\left(\frac{1}{1+g_{pt+1}}\right)\right]\begin{cases}<1, \text{ binding CIA constraint} \\ =1, \text{ nonbinding CIA constraint}\end{cases} \tag{16}$$

The term $[1/(1+g_{pt+1})]$ is the gross return of money, $R^M_{t+1} \equiv 1 + r^M_{t+1}$.[19] The left hand side of the above condition can also be written as $E_t\left[\psi_{t+1}R^M_{t+1}\right]$, where ψ_{t+1} is the stochastic discount factor or pricing kernel which is equal to the intertemporal rate of substitution (IRS) between next period consumption and current consumption. The term on the left hand side of (16) is the expected return of money measured in next period's utility per unit of current utility. When the expected return of a unit of money measured in next period's utility units is the same as the value of a unit of money measured in current utility units (i.e $E_t\left[\psi_{t+1}R^M_{t+1}\right] = 1$), the CIA constraint does not bind because the agents are indifferent between spending a unit of money today and holding it for one period. On the other hand, when the expected return of a unit of money, measured in next period's utility units, is smaller than the current utility value of a unit of money (i.e $E_t\left[\psi_{t+1}R^M_{t+1}\right] < 1$), the CIA constraint binds because agents are not indifferent between spending a unit of money today and holding it for one period; they strongly prefer to spend it today. Equivalently, when consumers expect that the *relative* value of money will decrease, they spend all their money holdings the current period and the CIA constraint binds, otherwise they keep some cash

[18]This is the same condition governing money demand in Alvarez, Atkeson and Edmond (2009). In their model, the condition holds with strict equality when the household carries a strictly positive balance of money in its bank account into next period. The latter is equivalent to a non-binding CIA constraint in our model. Note that using the logarithmic utility function, the left hand side of (16) can also be written as $\beta E_t[P_t c_t / P_{t+1} c_{t+1}] = \beta E_t[1/(1+g_{pt+1})(1+g_{ct+1})]$ where g_c denotes the growth rate of consumption.

[19]Note that $r^M = -g_p/(1+g_p)$ is non-positive as long as inflation is strictly non-negative.

for next period and the CIA constraint does not bind.

Dividing (13) over (14) using (3) yields:

$$\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} = \frac{\lambda_{1t}}{\lambda_{1t} + \lambda_{2t}} \frac{x'_{it}\left[p'(x_{it})\, x_{it} + p(x_{it})\right]}{P_t m} \tag{17}$$

or

$$\phi \frac{c_t}{1 - h_t} = \frac{\lambda_{1t}}{\lambda_{1t} + \lambda_{2t}} w_t \tag{18}$$

When $\lambda_{2t} = 0$ and the CIA constraint for that period is not binding or weakly binding, this is the usual intra-temporal condition which states that the marginal rate substitution (MRS) between leisure and consumption equals the real wage. However, when the CIA constraint is strictly binding with $\lambda_{2t} > 0$, the MRS is lower than the real wage, so that for given consumption the labor supply h_t is lower.[20] Consumption will be lower as well when $\lambda_{2t} > 0$ (the income effect) which will tend to increase h_t, but since the real wage remains constant the overall effect on the labor supply is negative. One way of understanding the leisure-consumption distortion when the CIA binds is that the household switches from consumption which is constrained by CIA to leisure which is not: the CIA in effect acts as a tax on consumption.

We can see that the behavior of the household divides into two regimes. In one regime (CIA constraint non-binding or weakly-binding) $\lambda_{2t} = 0$ and the household behaves in the standard way (it can demand and supply as much as it wants to at market prices and wages). In the other regime $\lambda_{2t} > 0$, the household is constrained in its ability to consume at the

[20] Condition (18) can be rewritten as $h_t = 1 - (1 + \lambda_{2t}/\lambda_{1t})(\phi/w_t)c_t$ so that for a given level of consumption, labor supply is lower when $\lambda_{2t} > 0$.

prevailing price: it would like to consume more given the price, but is unable to do so. This is an *effective demand constraint*: with a CIA constraint, the *desired* consumption can only become *effective* if there is the cash to execute it. This spills over into the labor supply decision, reducing the level of labor supply. There is less incentive to work now and increase income which cannot be spent this period only to generate more cash for next period when it is not needed. This is a very "Keynesian" effective demand mechanism, as was found in the earlier literature on non-Walrasian equilibria.[21]

2.3 Equilibrium with an Occasionally Binding CIA Constraint

There is a sequence of productivity levels and money supplies $\{\theta_t, M_t\}_{t=1}^{\infty}$ that evolve according to ϑ and ξ and the initial conditions $\{\theta_1, M_1\}$. Whilst we have treated q_t as given at the household level, we now need to define the aggregate relationship which determines the velocity of circulation:

Assumption *Let us define a velocity shock φ_t which has an initial condition φ_1 and the conditional p.d.f. $\bar{\Phi}(\widetilde{\varphi}; \varphi')$ for $\widetilde{\varphi} \in \Phi \subset \Re_+$ where φ' denotes the previous period realized value of φ. The velocity of circulation is determined by the mapping: $q_t \in Q_t$:* $\mathbb{Z}_+$ x Θ x Φ x $N \to (0, q^b]$ which we can write as $q_t = q(t, \theta_t, \varphi_t, \nu_t)$.

Thus we allow for a very general set of possibilities about how the velocity is determined: there is a general function (which may be time specific) which relates the velocity q_t to the two shocks determining θ_t, M_t as well as a possible velocity-specific shock. The assumption

[21] See for example Clower (1965), Leijonhufvud (1968), Benassy (1975), Malinvaud (1975). However, unlike these older papers, the phenomenon in the present model is very much dynamic and intertemporal rather than resulting from static and ad hoc rationing constraints that arise from exogenous fixed prices..

allows for the velocity to be constant, or to be decreasing or increasing in its arguments and there is no requirement for smoothness or differentiability. An equilibrium consists of a sequence pairs of $\{w_t,\ P_t\}_{t=1}^{\infty}$ that clear the labor and the goods market (recall that w is the *real wage* and P is the *nominal price* of output) given the economic fundamentals $\{\varphi_t, \theta_t, M_t\}_{t=1}^{\infty}$. Associated with $\{w_t,\ P_t,\ \theta_t,\ \varphi_t,\ v_t\}_{t=1}^{\infty}$ are the sequences $\{q_t,\ \lambda_{1t},\ \lambda_{2t},\ y_t,\ c_t,\ h_t,\ \pi_t\}_{t=1}^{\infty}$.

We can characterize the equilibrium sequence by dividing it into two possible states: one where the CIA is binding, and one where it is not. Of course, how this divides up will depend on the sequence of productivity, monetary and velocity-specific shocks. The two extremes are that the CIA constraint is always binding (as in Cooley and Hansen, 1989), or never binding. The following propositions allow us to determine how the economy behaves in the case of an intermittently binding CIA constraint.

For all t, the real wage is related to the current productivity level by the markup equation (8): $w_t = \theta_t(\gamma - 1)/\gamma$. The nominal price P_t thus becomes the key variable for establishing equilibrium in each period. A useful way to sort the sequence into binding and non-binding is to note that there is an upper bound to the velocity of circulation: the CIA constraint binds only when this upper bound is reached.

Proposition 1 *For all t there is an upper bound $q^b = \gamma/(\gamma - 1)$ on the equilibrium q_t. The CIA constraint binds at time t when $q_t = q^b$ and it does not bind at time t when $q_t < q^b$.*[22]

[22] Recall that whether the CIA constraint binds or not depends on the expectation about next period's *relative* value of money (condition (16)). This expectation is conditional on the current state of the economy.

All proofs are in the appendix. The intuition behind Proposition 1 is clear. Firstly, the upper bound on the velocity comes from two sources: the CIA constraint (12) itself, and the proportion of expenditure which is not subject to the CIA constraint (the expenditure of entrepreneurs which equals profits). Turning to the CIA constraint, if there were no profits (γ very large) then worker-household consumption equals output and (12) becomes $P_t y_t \leq M_t^c + \nu_t$, which implies by definition of the velocity (1) that $q^b = 1$. However, the entrepreneurs spend all of their profits and are not subject to the CIA. This means that the CIA constraint only applies to that portion of output which is consumed by workers. A higher markup implies a greater share of profits, and thus for a given output a lower share of consumption by worker-households and hence a higher overall velocity is possible. For γ close to 1, profits take up nearly all output and the CIA constraint only applies to a very small proportion of output, which allows the velocity to be very large. If the CIA constraint applied both to workers and entrepreneurs, then the share of profits would not matter and we would have no dependence of velocity on $\gamma : q^b = 1$. However, it seems more reasonable to assume that entrepreneurs are not so constrained. Hence, this "profit share effect" means that the upper bound of q_t is decreasing with the elasticity of demand of the consumption good (i.e $dq^b/d\gamma = -1/(\gamma - 1)^2 < 0$) or equivalently, it is increasing with the markup of marginal productivity over the real wage.

Proposition 1 enables us to partition time into two sets: times when the CIA constraint is strictly binding, and times when it is not binding:[23]

$$\mathcal{B} = \left\{ t \in \mathbb{Z}_+ : \ \lambda_{2t} > 0 \text{ and } q_t = q^b \right\}; \ \mathcal{NB} = \left\{ t \in \mathbb{Z}_+ : \lambda_{2t} = 0 \text{ and } q_t \leq q^b \right\}$$

Now, we can define the *proportion* of periods in which the CIA constraint is binding. If we define for any $T \in \mathbb{Z}_+$

$$\mathcal{B}(T) = \{ t \in \{1, 2...T\} : t \in \mathcal{B} \}$$

and likewise $\mathcal{NB}(T)$, we can define the proportion of times the CIA constraint binds until T:

$$\mathcal{P}(\mathcal{B}, T) = \frac{\#\mathcal{B}(T)}{T}$$

The stationarity of the conditional distributions of θ, ν and φ is sufficient to ensure that $\lim_{T \to \infty} \mathcal{P}(\mathcal{B}, T) = \varkappa$, where $\varkappa \in [0, 1]$. The following Propositions characterize the equilibrium price level P_t when the CIA binds and when it does not, and show that for given fundamentals, the proportion of time in which the CIA binds is non-decreasing in γ.

Proposition 2 (i) *When the CIA constraint binds* $(t \in \mathcal{B})$ *there is a unique equilibrium where* $P_t = (1 + \chi_t)\, q^b \left[\frac{M_t + \nu_t}{\theta_t} \right]$ *with* $\chi_t = \frac{\phi}{Z_t (M_t + \nu_t)} > \phi > 0$ *and* $Z_t = \beta E_t \left\{ \frac{u_c(c_{t+1}, l_{t+1})}{P_{t+1}} \right\}$.

(ii) *When the CIA constraint does not bind* $(t \in \mathcal{NB})$ *there is a unique equilibrium where* $P_t = (1 + \phi)\, q_t \left[\frac{M_t + \nu_t}{\theta_t} \right]$ *with* $q_t \leq q^b$ *and* $\chi_t \leq \phi$.

[23] Weakly binding and non-binding equilibria belong to the same category. Whenever we refer to binding CIA constraints we imply the strictly binding case.

The interpretation of Proposition 2(i) is that the CIA constraint binds when the expected return on savings is sufficiently low. Note that Z_t is the discounted expected marginal utility that \$1 saved now can buy next period:

$$Z_t = \beta E_t \left\{ \frac{u_c\left(c_{t+1}, l_{t+1}\right)}{P_{t+1}} \right\} = \beta E_t \left(\frac{1}{P_{t+1} c_{t+1}} \right)$$

When Z_t is low, and hence χ_t is high, the return to saving is so low that the worker-household wants to spend all of its cash balances now. The CIA constraint prevents the worker-household from borrowing to smooth its consumption as much as it would like to. The critical value of $\bar{Z}_t$ at which the CIA binds is defined in the following corollary:

Corollary 1 *Let* $\bar{Z}_t = \left(M_t + \nu_t\right)^{-1}$. *The CIA constraint strictly binds at time* t *when* $Z_t < \bar{Z}_t$ *(and hence* $Z_t\left(M_t + \nu_t\right) = \phi/\chi_t < 1$*), and does not bind when* $Z_t \geq \bar{Z}_t$*, (and hence* $Z_t\left(M_t + \nu_t\right) = q^b/q_t \geq 1$*).*

$\bar{Z}_t$ is the return on savings that exactly equates the marginal utility of current consumption to the expected discounted marginal utility of next-period consumption when the household spends all of its current money balance.[24] If Z_t falls below this critical level, then the CIA constraint binds and the worker-household is prevented from lowering its marginal utility of current consumption by increasing its current consumption. It is clear that this is an intertemporal phenomenon which depends on expectations about what is going to happen next period: indeed, since the CIA constraint can bind in the future it may involve expectations into the infinite future.

[24] With logarithmic utility, $u_c(c_t) = \left(M_t + \nu_t\right)^{-1}$ when all current balances are spent.

Corollary 1 also indicates that the velocity of circulation is related to the expectations about the future state of the economy via Z as $q_t = q^b / Z_t (M_t + \nu_t)$. Since a current change in money supply (ν_t) affects expectations about the future value of money (Z_t), velocity can be constant, increasing or decreasing in money supply. The direction of the effect of ν_t on q_t depends on how changes in money supply affect expectations. This is consistent with our assumptions about the functional form of velocity. For instance, if an expansionary money supply generates expectations for a decrease in the value of money next period, then it is possible that an increase in ν causes an increase in velocity.

Corollary 2 *When $t \in \mathcal{NB}$ and the CIA constraint weakly binds then, $Z_t (M_t + \nu_t) = 1$, $\phi = \chi_t$ and $q_t = q^b$.*

The implications for the CIA constraint on nominal prices and real output can be seen if we rewrite the expression for the price level using the explicit functional forms:

$$P_t = \left[\frac{q^b}{\theta_t} \right] \left[(M_t + \nu_t) + \frac{\phi}{Z_t} \right] \quad \text{for } t \in \mathcal{B}$$

The equilibrium price level is not proportional to the current money-supply $M_t + \nu_t$ due to expectations $\phi / Z_t > 0$. To show this let $\nu_t = \alpha(\nu_t) M_t$ and $Z_t \in \left[\underline{\zeta}, \zeta, \overline{\zeta} \right]$ such that $0 < \underline{\zeta} < \zeta < \overline{\zeta}$ and $\mu(\nu_t)$ to denote the percentage effect of ν_t on P_t. If $\eta(\nu_t)$ is the percentage effect of ν_t on ϕ / Z_t such that $\eta(\nu_t) \in \left[\underline{\eta}, 0, \overline{\eta} \right]$ with $\underline{\eta} < 0 < \overline{\eta}$ and $\phi / \overline{\zeta} = \left(1 + \underline{\eta}\right) \phi / \zeta < \phi / \zeta < (1 + \overline{\eta}) \phi / \zeta = \phi / \underline{\zeta}$ then

$$\mu(\nu_t) = \alpha(\nu_t) \frac{M_t}{M_t + \frac{\phi}{\zeta}} + \eta(\nu_t) \frac{\frac{\phi}{\zeta}}{M_t + \frac{\phi}{\zeta}} \quad \text{for } t \in \mathcal{B}$$

Even if a change in money supply does not affect expectations (i.e $\eta(\nu_t) = 0$), $\mu(\nu_t) < \alpha(\nu_t)$ because $\phi/Z_t > 0$. In other words, a 10% higher money-supply implies a higher price, but one which is less than 10% higher. When a change in money supply leads to expectations for higher *absolute* value of money (i.e $\eta(\nu_t) = \underline{\eta}$) then the percentage increase in the price level, $\mu(\nu_t)$, is even smaller than in the case of $\eta(\nu_t) = 0$. Note that if $(t-1) \in \mathcal{B}$ then $\mu(\nu_t)$ is the time t inflation rate which is due to the change in money supply. Therefore, there are cases where the price level responds sluggishly to a change in money supply.

Proposition 2 also indicates that the binding CIA constraint implies a non-neutrality of money. It is straightforward to show that output and consumption respond *negatively* to the CIA constraint (see proofs of Propositions 1 and 2):

$$y_t = \frac{nm}{1+\omega_t}\theta_t,\ c_t = \frac{y_t}{nmq^b},\ h_t = \frac{1}{1+\omega_t},\ \ \pi_t = \frac{y_t}{n\gamma}$$

where $\omega_t = \chi_t$ for $t \in \mathcal{B}$ and $\omega_t = \phi$ for $t \in \mathcal{NB}$. The strength of the CIA constraint is reflected in how big χ_t is (since it is inversely related to Z_t). In the absence of CIA constraint, when $Z_t = \bar{Z}_t$ from proposition 2(ii) and corollary 1, we have $\chi_t \leq \phi$; when the CIA constraint binds we have $\chi_t > \phi$. Hence, output, employment and profits are all lower with a binding CIA constraint than without. This is intuitive, since the restriction of consumption directly reduces output and hours per worker (from the production function and labour market equilibrium) and profits (via the markup equation 10). Hence, if we compare outputs in times with the non-binding constraint (where output is at its efficient level y_t^*)[25] and when it is binding we have:

$$y_t = \frac{nm}{1+\phi}\theta_t = y_t^* \text{ for all } t \in \mathcal{NB} \text{ and } y_t = \frac{nm}{1+\chi_t}\theta_t < y_t^* \text{ for all } t \in \mathcal{B}$$

If we compare any two periods with the same productivity level, we can say that the non-binding equilibrium Pareto dominates the binding equilibrium in terms of the current flow in utility and profits. Furthermore, we can say that if we have two periods with the same productivity in which the CIA constraint binds, the one with the smaller χ_t dominates the other.

Proposition 3 *(i) For any $t_1 \in \mathcal{B}$ and any $t_2 \in \mathcal{NB}$* such that $\theta_{t_1} = \theta_{t_2}$ then $u(\theta_{t_2}) > u(\theta_{t_1})$ *and* $\pi(\theta_{t_2}) > \pi(\theta_{t_1})$. *(ii) For any $t_1, t_2 \in \mathcal{B}$* such that $\theta_{t_1} = \theta_{t_2}$, if $\chi_{t1} > \chi_{t2}$ then $u(\theta_{t_2}) > u(\theta_{t_1})$ *and* $\pi(\theta_{t_2}) > \pi(\theta_{t_1})$.

The role of imperfect competition matters in this model because entrepreneurs are assumed to be unaffected by the CIA constraint. The proportion of expenditure in the economy covered by the CIA constraint is increasing in the elasticity of demand (decreasing in the markup). We can now consider two economies that are identical in terms of the economic fundamentals over time, but which differ in the degree of imperfect competition. We can show that the CIA constraint cannot bind for a lower proportion of the time in a more competitive economy.

[25] Note that with the utility function assumed for the worker-consumer, the income and substitution effects of the real wage exactly offset each and equilibrium labour supply is unaffected by the degree of imperfect competition or productivity. In this case, it is only the CIA constraint that can alter employment and reduce output below its efficient level.

Proposition 4 *Consider* γ_1 *and* γ_2 *with corresponding sequences of equilibria and* resultant $\varkappa_1$ *and* $\varkappa_2$. *If* $\gamma_1 > \gamma_2$, *then* $\varkappa_1 \geq \varkappa_2$.

As the market becomes more competitive (as $\lim_{\gamma \longrightarrow \infty} q^b = 1$), it is "more likely" that the CIA constraint will bind (or certainly no less likely). It needs to be stressed that Proposition 4 does not imply that in a perfectly competitive market the CIA constraint will always bind. Whilst it is possible that the CIA constraint will be binding all the time and $\mathcal{NB} = \varnothing$, it is also perfectly possible (see Proposition 10(ii)) that in the competitive case the CIA constraint may never (strictly) bind and hence $\mathcal{B} = \varnothing$.[26] However, what is clear from the proof of Proposition 4 is that for some pairs (γ_1, γ_2), $\varkappa_1 < \varkappa_2$.

Proposition 4 implies that as the market becomes more competitive, it becomes "more likely" that output will be lower than its efficient level. Although this may sound counter-intuitive, it is justified by the presence of the CIA constraint which affects the portion of consumption being subject to the CIA constraint. As the elasticity of demand (γ) increases, firms face tougher competition, and the markup they charge reduces (i.e the monopoly power of the firms decreases). Firm owners are worse off by increased competition because (i) their share in aggregate production decreases and (ii) aggregate production is lower than its efficiency level when the CIA constraint binds. On the contrary, worker-consumers face a tradeoff between lower output when the CIA constraint binds and increased share in aggregate production. When the latter dominates the former, worker-consumers are better off from increased competition.

[26] Cooley and Hansen (1989), assume that the consumption good is traded in a perfectly competitive market. They establish the condition under which the CIA constraint binds and they assume that this condition is met at all times. This condition is a version of the condition established in Corollary 1.

We now show that monetary policy depends on the degree of competition. Two economies characterized by different degrees of competition but identical in all other respects will have different monetary policies, $\{M_t\}_{t=0}^{\infty}$, unless they have different expectations about the evolution of money supply, ξ. For simplicity we have assumed that the transition probabilities of money transfers depend only on the previous realization of the transfer. However, this assumption does not play a crucial role and the analysis can be easily extended when the transition probabilities have a more complex functional form and/or depend on other factors. Let Ξ denote the conditional cumulative distribution of ξ. Then, the following proposition holds.

Proposition 5 *If for any ν_a and ν_b such that when $\nu_a < \nu_b$, $q(\nu_a) < q(\nu_b)$ then, for a given sequence $\{\theta_t, M_t, \varphi_t\}_{t=1}^{\infty}$, probability distributions ϑ and $\overline{\Phi}$, and γ_1 and γ_2 with corresponding cumulative distributions Ξ^1 and Ξ^2 such that $\gamma_1 > \gamma_2$: (i) when $t(\gamma_1) \in \mathcal{NB}$ then $t(\gamma_2) \in \mathcal{NB}$ and Ξ^1 first-order stochastically dominates Ξ^2, (ii) when $t(\gamma_1) \in \mathcal{B}$ then $t(\gamma_2) \in \mathcal{B}$ and Ξ^2 first-order stochastically dominates Ξ^1.*

In section 3, we analyze the case of perfect foresight and show (proposition 7) that for low (negative) growth rates of money supply the CIA constraint does not bind whereas whenever the CIA constraint binds the growth rates of money supply is above a certain threashold. This motivates the assumption that velocity is an increasing function of money transfers. If this is the case, then for a given sequence $\{\theta_t, M_t, \varphi_t\}_{t=1}^{\infty}$ and probability distributions ϑ *and* $\overline{\Phi}$, as the market becomes more competitive, it is relatively more likely that the growth rate of money will increase when the CIA constraint does not bind and relatively less likely that the growth rate of money will increase when the CIA constraint binds. Monetary policy is optimal in the sense that given transition probabilities, the sequence of money transfers is such that it satisfies the households' and firms' optimal conditions. If the transition probabilities for money transfers are the same in the two economies with different degrees of imperfect competition then, for each economy there will be a different sequence of money transfers satisfying the optimal conditions.

2.4 Discussion

Propositions 1-5 show that in this simple economy, we can divide time into two regimes. In one, where the CIA constraint does not bind, we live in a Classical world where real variables are given by their optimal level (conditional on current productivity and the presence of monopolistic competition), prices adjust instantaneously to current shocks in velocity, productivity or the money supply. In the other regime, the CIA constraint binds, and output falls below its optimal level. Households see the expected marginal utility of their money holdings falling to a very low level in the next period: perhaps they expect a high nominal price next period (or a productivity boom) and would like to increase their current consumption to lower their current marginal utility. However, they run into the CIA constraint: markets clear, but at a lower level of output and consumption. The nominal price that equates the cash-constrained demand with the supply is higher than in the classical regime (Proposition 2(i)). Prices are perfectly flexible, but in this Keynesian regime where the CIA binds there is an effective demand effect: *the price-level itself influences the way the CIA operates.*[27] In essence, there are two forces operating in response to the low value of

[27] When the household is operating under a CIA constraint, its demand curve becomes a rectangular hyperbola rather then the normal demand.

expected marginal utility per \$ next period: on the one hand, the current price rises to reduce the current marginal utility per \$, on the other hand the households are trying to increase their consumption. Since the CIA constraint prevents them from increasing consumption enough, the equilibrium market clearing nominal price is higher than it would have been in the absence of the constraint.

Why does not the price adjust downward to avoid the CIA effect and let the household raise its current consumption sufficiently? The answer is in the general equilibrium: the maximum output that the economy can produce under voluntary trade is given by y_t^*. With a lower price than that given by Proposition 2(i), the demand of the consumer would exceed the supply. With the lower prices the worker-household would be wanting to consume more than it was willing to produce through supplying its own labour (given that a proportion of output goes to entrepreneurs). So higher current prices are consistent with both the current equilibrium in goods and labor markets, and also ensure that the inter-temporal equilibrium holds given the CIA constraint.

To make matters concrete, for illustrative purposes, let us assume that the velocity of circulation is an increasing function of θ and ν.[28] This assumption is not short of empirical support: Chiu (2007), provides evidence for the positive relationship between velocity and money while Hromcová (2008) provides evidence for the positive relationship between veloc-

[28] This is a special case of a velocity function where $q_t = q(\underset{+}{\theta_t}, \underset{+}{\nu_t})$. Alvarez et al. (2009), provide evidence that the correlation between measures of money and velocity is negative. However, this does not necessarily imply that money supply is the dominant factor that drives velocity. This can be illustrated beyond the context of the current model. For instance, suppose technology is the dominant factor of velocity and that it affects it positively. Then, if technology deteriorates, it is reasonable to assume that the monetary authority increases the supply of money to boost the economy. In this case, even though money transfers affect velocity positively, overall money supply and velocity exhibit a negative correlation. Therefore, by just looking at correlations between money and velocity we cannot safely draw conclusions about the relationship between velocity and transfers.

ity and quality of technology in production. It follows that for a massive monetary expansion or a substantial technology improvement or a combination of the two, the CIA constraint will then bind because the agents expect that the value of money next period will be smaller than the value of money the current period (see condition (16)). As a result, they rush to spend all their money holdings the current period which increases the velocity of money to the extent that it hits its upper bound. Then, equilibrium output, consumption, work effort and profits, all depend on the current money supply as well as expectations for future money transfers, technology innovations and velocity-specific shocks.

In general, a higher level of technology would imply a higher welfare. In addition, for any given technology level, a binding CIA constraint implies a lower welfare than a non-binding CIA constraint (Proposition 3). A higher level of technology would also imply a higher probability of a binding CIA constraint (under our illustrative assumption). If the CIA constraint binds, larger money transfers will, in general, increase the welfare. The monetary effect on real quantities comes through variable χ. The smaller χ is the higher the welfare of both consumers and firm owners. There are two channels through which money transfers can affect χ, a direct channel in which there is a negative relationship between ν and χ, and an 'indirect' channel (through Z) in which the direction of the relationship is not obvious because it depends on the expectations of consumers about next period's value of money. The latter depends on the conditional probability distributions of ν, θ and φ. Assuming that the direct effect of ν on χ dominates the indirect effect, an increase in the supply of money decreases χ and thereby, increases welfare along a binding CIA constraint.

Note that when the monetary authority decides the transfer ν_t, the values of θ_t and φ_t

are not known. For a given technology innovation and velocity-specific shock the monetary authority can increase the likelihood of a binding CIA constraint by transferring a large amount of money to the agents. A binding CIA constraint can occur even with moderate levels of technology. If such a case occurs then, according to Proposition 3, the welfare for both firm owners and consumers will deteriorate.[29] The monetary authority cannot entirely prevent the CIA constraint from binding because the condition that determines a binding CIA constraint does not depend only on ν but also on θ and φ, which are not under the control of the monetary authority. One may argue that the monetary authority should keep money supply constant, making zero transfers, in order to decrease the likelihood of a binding CIA constraint. Variation in the supply of money however does not necessarily make the consumers worse off. As mentioned above, there might be values of ν (within the set of equilibria with binding CIA constraints) that make the agents better off. In the absence of velocity shocks, if there was no time lag between the decision of the transfer and the realization of technology innovation then the monetary authority could have made appropriate transfers so that the agents achieve the highest level of welfare for any realization of θ. Furthermore, due to the time lag between decision from the monetary authority and consumers receiving the transfer as well as other possible frictions there is no guarantee that the full amount of the transfer as decided by the monetary authority will reach the consumers. Even if the monetary authority commits to a certain sequence of transfers, the uncertainty that consumers have about the transfers exists and is justified. Consequently, in a stochastic environment, the monetary authority cannot achieve with certainty a non-

[29] If the CIA constraint did not bind utility and real profits would have been higher at the same level of technology.

binding CIA constraint.

An example of welfare improving expansionary money supply

For simplicity, we abstract from velocity-specific shocks and assume that velocity is time invariant (i.e $q_t \in Q_t$: N x $\Theta \rightarrow (0, q^b])$. Let $\boldsymbol{\theta} = [\theta_1,\, \theta_2,\, \theta_3]' \in \Re^3_+$ and $\boldsymbol{\nu} = [\nu_1,\, \upsilon_2,\, \nu_3] \in N^3$ be vectors containing the possible values of θ and ν, respectively. Specifically, $\theta_1 < \theta_2 < \theta_3$ and $\nu_1 < \upsilon_2 < \nu_3$. The 3 x 3 transition matrices of $\boldsymbol{\theta}$ and $\boldsymbol{\nu}$ are denoted by $\boldsymbol{\vartheta}$ and ξ, respectively. Consider the following case:[30]

state	$\theta_3,\, \nu_1$	$\theta_3,\, \nu_2$	$\theta_3,\, \nu_3$	$\theta_2,\, \nu_1$	$\theta_2,\, \nu_2$	$\theta_2,\, \nu_3$	$\theta_1,\, \nu_1$	$\theta_1,\, \nu_2$	$\theta_1,\, \nu_3$
CIA const. binds	yes	yes	yes	no	no	yes	no	no	no

Notice that having a high ν increases the likelihood of a binding CIA constraint which is a welfare inferior outcome as regards to the current welfare of the agents for any given level of technology. Nevertheless, the monetary authority will not necessarily choose a low value of ν in order to decrease the probability of a binding CIA constraint. If θ_3 occurs the CIA constraint will bind no matter what ν is. Then, it could be the case that among the binding CIA-constraint equilibria, $\chi(\theta_3, \nu_3) < \chi(\theta_3, \nu_1)$ where $\nu_1 < \nu_3$. The latter implies that both current profits and current utility are higher under ν_3 than under ν_1; as shown in the proof of proposition 5: $du/d\chi < 0$ and $d\pi/d\chi < 0$. Whether $\chi(\theta_3, \nu_3)$ is smaller than $\chi(\theta_3, \nu_1)$ depends on the expectations of consumers about the future value of money (Z). Consequently, there might be a scenario where there is a trade off between choosing a low

[30]Suppose the economy is at state (θ_k, ν_f) then, $Z_t(\theta_k, \nu_f) = \beta \sum_{j=1}^{3} \sum_{i=1}^{3} \vartheta_{kj} \xi_{fj} \frac{q^b}{q(\theta_i, \nu_j)} \frac{1}{M_t + \upsilon_f + \nu_j}$, where ϑ_{ij} and ξ_{ij} are the ith, jth elements of matrices $\boldsymbol{\vartheta}$ and ξ, respectively. Note also that q_t is such that $q_t = q^b/Z_t(M_t + \nu_t)$ when the CIA constraint binds and $q_t = q^b$ when the CIA constraint does not bind.

value of ν that reduces the probability of a binding CIA constraint and a high value of ν that increases welfare among binding CIA-constraint equilibria.

3 The Special case of Perfect Foresight

In this section, we analyze the case where agents have perfect foresight. With perfect foresight, there is no role for money as a buffer-stock: its only potential role is as a store of value and medium of exchange. Whilst this is very much a simple and special case, we can see how the framework we have set up can shed light on the possibilities contained in Propositions 1-2. Firstly, we will define a zero inflation steady state. For a steady-state to be possible, we have to assume that there are no shocks: $\theta_t = \widehat{\theta}$, $\nu_t = 0$, $\varphi_t = \widehat{\varphi}$. Given there are no shocks, all real and nominal variables are assumed constant.

Definition of zero-inflation steady state: *For* $\{\theta_t = \widehat{\theta},\ \nu_t = 0,\ \varphi_t = \widehat{\varphi}\}_{t=1}^{\infty}$, $q_t = \widehat{q}$, $\lambda_{1t} = \widehat{\lambda}_1$, $\lambda_{2t} = \widehat{\lambda}_2$, $y_t = \widehat{y}$, $c_t = \widehat{c}$, $h_t = \widehat{h}$, $w_t = \widehat{w}$, $M_t = \widehat{M}$, $P_t = \widehat{P}$ *and* $\pi_t = \widehat{\pi}$.

Proposition 6: *At the zero-inflation steady state, when* $\beta \in (0,1)$, *the CIA constraint always strictly binds, with* $\widehat{\lambda}_2 > 0$, $\widehat{q} = q^b$, *and* $\widehat{P} = [1 + \phi(2-\beta)]\,\widehat{q}\left[\frac{\widehat{M}}{\widehat{\theta}}\right]$. *Then, real variables are given by:* $\widehat{y} = \frac{nm}{1+\phi(2-\beta)}\widehat{\theta}$, $\widehat{c} = \frac{\widehat{y}}{nmq^b}$, $\widehat{h} = \frac{1}{1+\phi(2-\beta)}$, $\widehat{\pi} = \frac{\widehat{y}}{n\gamma}$.

So, in steady-sate with zero-inflation no one will want to hold money at the end of the period. Since consumption is constant, the discounted marginal utility of consumption next period is always less than current marginal utility, so that with a zero rate of return on money holdings, a \$ today will always buy more utility than a \$ tomorrow. This implies that the velocity of money will always be at its upper bound (since there are no velocity shocks, this is constant). The level of output in steady-state is less than would occur when the CIA is non-binding, but only very slightly. The ratio of steady-state output and employment to the efficient level is:

$$\frac{\widehat{y}}{y^*} = \frac{\widehat{h}}{h^*} = \frac{1+\phi}{1+\phi+\phi(1-\beta)} < 1$$

Clearly, if we are dealing with quarterly data, then $\beta = 0.995 \approx 1$ and the ratio is close to unity. For example, with $\phi = 1$, this level of discounting gives us a ratio of 0.9975 (4 s.f.). This slight inefficieny is caused by the distortion of the work-leisure decision that occurs when the CIA constraint binds: the consumption-leisure MRS is less than the real wage, so that the supply of labour is lower (for a given level of consumption). To see why the CIA constraint needs to strictly bind, assume instead that it was weakly binding with $\widehat{q} = q^b$ and $\hat{\lambda}_2 = 0$: in this case, the household could increase its utility by bringing forward some consumption (since $\beta < 1$) and hence the steady-state is only sustainable with $\hat{\lambda}_2 > 0$.

Now we consider the general case where consumer-households and firm-owners perfectly foresee the evolution of the economic fundamentals $\{\varphi_t, \theta_t, \nu_t\}_{t=1}^{\infty}$.[31] Let $g_{jt} = (j_t/j_{t-1}) - 1$ denote the growth rate of variable j at time t. We turn first to the growth rate of the nominal money supply.

Proposition 7: *In the economy with perfect foresight*: (i) *when* $t \in \mathcal{B}$ *then* $g_{Mt+2} > \beta - 1$ *but the reverse does not always hold,* and (ii) *when* $g_{Mt+2} \leq \beta - 1$, *then* $t \in \mathcal{NB}$ *but the reverse does not always hold.*

[31] The analysis of the perfect foresight equilibrium can be generalized to the case of non-stable state variables.

Cooley and Hansen (1989, p. 736), argue that in their model $g_{Mt+2} > \beta - 1$ is a sufficient condition for the CIA constraint to be always binding. In our model, $g_{Mt+2} > \beta - 1$ is not a sufficient condition for the CIA constraint to be always binding due to the fact that velocity is allowed to vary. Note that conditions $g_{Mt+2} > \beta - 1$ and $g_{Mt+2} \leq \beta - 1$ can be rewritten as $g_{\nu t+1} > [(M_t + \nu_t)/\nu_t](\beta - 1) - 1$ and $g_{\nu t+1} \leq [(M_t + \nu_t)/\nu_t](\beta - 1) - 1$, respectively.[32] The two conditions can also be written as $\nu_{t+1} > (M_t + \nu_t)(\beta - 1)$ and $\nu_{t+1} \leq (M_t + \nu_t)(\beta - 1)$, respectively. Since $\beta \in (0, 1)$, the latter shows that both binding and non-binding CIA constraints are consistent with both positive and negative money transfers.

For any $T \in \mathbb{Z}_+ \cup \{0\}$, let us define the following sets

$$\widetilde{\mathcal{B}}(T) = \{t \geq T + 1 : t \in \mathcal{B}\}\,;\ \widetilde{\mathcal{NB}}(T) = \{t \geq T + 1 : t \in \mathcal{NB}\}$$

such that $\mathcal{B}(T) \cap \widetilde{\mathcal{B}}(T) = \varnothing$, $\mathcal{B}(T) \cup \widetilde{\mathcal{B}}(T) = \mathcal{B}$, $\mathcal{NB}(T) \cap \widetilde{\mathcal{NB}}(T) = \varnothing$, $\mathcal{NB}(T) \cup \widetilde{\mathcal{NB}}(T) = \mathcal{NB}$, $\mathcal{B}(0) = \varnothing$, $\mathcal{NB}(0) = \varnothing$, $\widetilde{\mathcal{B}}(0) \equiv \mathcal{B}$ and $\widetilde{\mathcal{NB}}(0) \equiv \mathcal{NB}$. In addition, let us define the following auxiliary sets

$$\mathcal{M}^{\leq}(T) = \{t \geq T + 1 : g_{Mt+2} \leq \beta - 1\}$$

$$\mathcal{M}^{>}(T) = \{t \geq T + 1 : g_{Mt+2} > \beta - 1\}$$

[32] Refer to the proof of proposition 7.

Then, using proposition 7 and its proof we can define the mutually exclusive sets $\widetilde{\mathcal{NB}}_1(T)$ and $\widetilde{\mathcal{NB}}_2(T)$:

$$\widetilde{\mathcal{NB}}_1(T) = \left\{t \in \mathcal{M}^{\leq}(T) : \ t \in \mathcal{NB}\right\}$$

$$\widetilde{\mathcal{NB}}_2(T) = \left\{t \in \mathcal{M}^{>}(T) : \ q_{t+1} < q_t \leq q^b, \ t \in \mathcal{NB}\right\}$$

where $\widetilde{\mathcal{NB}}_1(T) \cap \widetilde{\mathcal{NB}}_2(T) = \varnothing$ and $\widetilde{\mathcal{NB}}_1(T) \cup \widetilde{\mathcal{NB}}_2(T) = \widetilde{\mathcal{NB}}$.[33] The second part of proposition 7 indicates that if the growth rate of money is always less or equal than $\beta - 1$ from any $t^* \in \mathbb{Z}_+$ onwards, the CIA constraint will never bind again. The case of $\mathcal{M}^{>}(T) = \varnothing$ or $g_{Mt+2} \leq \beta - 1$ with $\beta \in (0,1)$ for all $t \geq T+1$ holds only if $g_{Mt+2} > -1$ for all $t \geq T+1$.[34] Therefore, when $\mathcal{M}^{>}(T) = \varnothing$, it must be that $-1 < g_{Mt+2} \leq \beta - 1$. Proposition 7$(i)$ also indicates that it is possible that $g_{Mt+2} > \beta - 1$ when $t \in \widetilde{\mathcal{NB}}(T)$ which occurs when $t \in \widetilde{\mathcal{NB}}_2(T)$.

Corollary 3 *In the economy with perfect foresight, for any* $\beta \in (0,1)$ *and any* $T \in \mathbb{Z}_+ \cup \{0\}$: (i) $\varnothing \subseteq \widetilde{\mathcal{B}}(T)$ *and* (ii) $\varnothing \subseteq \widetilde{\mathcal{NB}}(T)$.

Corollary 3 signifies that there are sequences of $\{\theta_t, \nu_t, \varphi_t\}$ such that (i) the CIA constraint never binds and (ii) the CIA constraint always binds. For $\widetilde{\mathcal{B}}(T) = \varnothing$, the sequence of money transfers, $\{\nu_t\}_{t=T+1}^{\infty}$, can be complemented by sequences of velocity and technology innovations, $\{\theta_t, \varphi_t\}_{t=T+1}^{\infty}$, such that $\widetilde{\mathcal{NB}}_2(T) \neq \varnothing$.

[33] These relationships do not necessarily hold in the stochastic model.
[34] If $g_{Mt+2} < -1$, the positivity of money supply will be violated.

Proposition 8 *In the economy with perfect foresight, there is a unique equilibrium for* P_t, y_t, c_t, h_t and π_t *such that*

$$P_t = (1+\xi_t)\, q_t \left[\frac{M_t+\nu_t}{\theta_t}\right] \quad with \quad \begin{cases} \xi_t = \chi_t,\ q_t = q^b \text{ and } \chi_t > \phi \text{ when } t \in \mathcal{B} \\ \xi_t = \phi,\ q_t \leq q^b \text{ and } \chi_t \leq \phi \text{ when } t \in \mathcal{NB} \end{cases}$$

$$y_t = \frac{nm}{1+\omega_t}\theta_t,\ c_t = \frac{y_t}{nmq^b},\ h_t = \frac{1}{1+\omega_t} \quad and \quad \pi_t = \frac{y_t}{n\gamma}$$

$$where\ \omega_t = \begin{cases} \chi_t \text{ for } t \in \mathcal{B} \\ \phi \text{ for } t \in \mathcal{NB} \end{cases} \quad with\ \chi_t = \begin{cases} \frac{\phi}{\beta}\left(1+g_{Mt+2}\right) \text{ for } t+1 \in \mathcal{B} \\ \frac{\phi}{\beta q^b}\left(1+g_{Mt+2}\right) \text{ for } t+1 \in \mathcal{NB} \end{cases}$$

Since $\chi_t > \phi$ when $t \in \mathcal{B}$, for a given technology level, a non-binding equilibrium Pareto dominates a binding equilibrium in terms of welfare for both firm-owners and household-consumers (proposition 3). Note that if the CIA binds in period t but is expected to be non-binding in $t+1$, the upper-bound on the q^b enters into χ_t. This implies that the degree of imperfect competition matters: a higher markup implies a higher q^b, which implies a higher output. Thus, a monetary authority which is interested in maximizing welfare, will choose the flow of money in every period such that the CIA constraint never binds. Corollary 3 indicates that this is possible since the binding set can be an empty set.

Corollary 4 *In the economy with perfect foresight, for any* $t \in \mathcal{NB}$, $g_{ct+1} \leq g_{\theta t+1}$.

Proposition 9: *In the economy with perfect foresight*: (*i*) *when* $t \in \mathcal{NB}$ *then* $g_{pt+1} \geq \frac{\beta}{1+g_{\theta t+1}} - 1$, *but the reverse does not always hold, and* (*ii*) *when* $g_{pt+1} < \frac{\beta}{1+g_{\theta t+1}} - 1$ *then* $t \in \mathcal{B}$, *but the reverse does not always hold.*

Corollary 4 indicates that whenever the CIA is non-binding, the growth rate of consumption next period cannot be greater than the rate of improvement in technology. As shown in the proof of proposition 9, when the CIA constraint binds, it is perfectly possible that the growth rate of consumption next period is greater than the rate of improvement in technology. This occurs because of an increase in work effort which boosts further the growth rate of production. In this case the gross inflation rate is smaller than $\beta/(1+g_{\theta t})$ due to the fact that $\chi_{t-1} \geq \chi_t$. From proposition 8, the latter also implies that not only output and consumption grow faster than the rate of improvement in technology but also real profits. As proposition 3 (*ii*) indicates, since $\chi_{t-1} \geq \chi_t$ neither household-consumers nor firm-owners are worse-off in the transition from period $t-1$ to period t.

Corollary 5 *In the economy with perfect foresight,* $(t-1) \in \mathcal{NB}$ *if and only if* $g_{pt} = \frac{\beta}{1+g_{ct}} - 1$*, otherwise* $(t-1) \in \mathcal{B}$*, and* $g_{pt} > \frac{\beta}{1+g_{ct}} - 1$ *: (i) If* $(t-1) \in \mathcal{NB}$ *and* $t \in \mathcal{NB}$ *then,* $g_{pt} = \frac{\beta}{1+g_{\theta t}} - 1$ *but the reverse does not always hold; (ii) If* $(t-1) \in \mathcal{NB}$ *and* $t \in \mathcal{B}$ *then,* $g_{pt} > \frac{\beta}{1+g_{\theta t}} - 1$ *but the reverse does not always hold; (iii) If* $(t-1) \in \mathcal{B}$ *then,* $g_{pt} > \frac{\beta}{1+g_{\theta t}} - 1$ *or* $g_{pt} \leq \frac{\beta}{1+g_{\theta t}} - 1$ *for any* t*.*

Corollary 5 (*i*) indicates that if the CIA constraint does not bind in two consecutive periods, the growth rate of the price level is a function only of the growth rate of technology. Under those circumstances, as technology improves prices must be falling. Corollary 5 (*i*) also demonstrates that if technology remains unchanged when the CIA constraint does not bind in two consecutive periods, prices decline at the rate $1-\beta$.

Corollary 6 *In the economy with perfect foresight, (i) If* $(t-1) \in \mathcal{NB}$ *and* $t \in \mathcal{NB}$ *then,* $g_{Mt+1} = \frac{\beta}{1+g_{qt}} - 1$ *but the reverse does not always hold; (ii) For any bundle* $(t-1)$ *and* t *other than* $\{(t-1) \in \mathcal{NB},\ t \in \mathcal{NB}\}$, $g_{Mt+1} > \frac{\beta}{1+g_{qt}} - 1$ *or* $g_{Mt+1} \leq \frac{\beta}{1+g_{qt}} - 1$.

Money growth on the other hand, along two consecutive non-binding CIA constraints, depends on the growth rate of velocity which is a function of the money transfer, technology and velocity innovation.[35] For, $\mathbb{Z}_+(T) = \{T+1, T+2, ..\infty\}$, it is also useful to partition time into periods of positive growth rates of technology and times of non-positive growth rates of technology:

$$\mathcal{G}^+(T) = \{t \in \mathbb{Z}_+(T) :\ g_{\theta_t} > 0\ \}\ ;\ \mathcal{G}^-(T) = \{t \in \mathbb{Z}_+(T) : g_{\theta_t} \leq 0\}$$

such that $\mathcal{G}^+(T) \cup \mathcal{G}^-(T) = \mathcal{G}(T)$. Corollary 5 indicates that for any $T \in \mathbb{Z}_+ \cup \{0\}$ and $\beta \in (0,1)$ such that $\widetilde{\mathcal{NB}}(T) = \mathbb{Z}_+(T)$, (i) if $\mathcal{G}^+ = \mathbb{Z}_+(T)$ then, $g_{pt} < 0$ for all t and (ii) if $\mathcal{G}^- = \mathbb{Z}_+(T)$ then $g_{ct} < 0$ for all t.

3.1 Inflationary steady-states and the optimal rate of inflation

We are now in a position to analyze non-zero-inflation steady-states, which we define as follows:

Definition of the inflationary steady-state *For* $\{\theta_t = \widehat{\theta},\ \nu_t = 0,\ \varphi_t = \widehat{\varphi}\}_{t=1}^{\infty}$, $q_t = \widehat{q}$,

$$\lambda_{1t} = \widehat{\lambda}_1,\ \lambda_{2t} = \widehat{\lambda}_2,\ y_t = \widehat{y},\ c_t = \widehat{c},\ h_t = \widehat{h},\ w_t = \widehat{w},\ \pi_t = \widehat{\pi},\ g_{Mt} = g_{pt} = \hat{g}_p.\ \textit{for all } t.$$

[35] If velocity is a continuously differentiable function in all arguments (technology level, money transfer and velocity innovation) then, $g_{qt} = \varepsilon_t^{q,\varphi} g_{\varphi t} + \varepsilon_t^{q,\nu} g_{\nu t} + \varepsilon_t^{q,\theta} g_{\theta t}$ where $\varepsilon_t^{q,i}$ is the elasticity of velocity with respect to variable i and $g_{\nu_t} = g_{Mt+1}(1+g_{Mt})/g_{Mt} - 1$. Then, using corollary 6(i), we can express g_{Mt+1} as a function of g_{Mt}, $g_{\varphi t}$, $g_{\theta t}$ and elasticities.

In the inflationary steady-state, money growth equals steady-state inflation and all real variables are constant.[36] The presence of steady-state inflation means that there is an inflation tax: holding money to finance transactions can incur a cost as prices are rising. This was of course implicit in Propositions 7-9. We can now state the following:

Proposition 10 Consider an inflationary steady-state:

(i) if $\hat{g}_p > \beta - 1$, then the CIA constraint always strictly binds, with real variables given by Proposition 7.

(ii) if $\hat{g}_p = \beta - 1$, then the CIA constraint never binds and the real variables are at the efficient levels defined in Proposition 2 (ii).

(iii) if $\hat{g}_p < \beta - 1$ then no steady-state exists.

Proposition 10(i) states that output is decreasing with the level of steady-state inflation: a higher inflation tax increases the distortion induced by the CIA constraint. If we define the welfare corresponding to a constant level of inflation as the per period flow of utility in the corresponding steady-state (and zero if there is no steady-state) then it follows that:

Corollary 7 *The optimal steady-state inflation rate is $\hat{g}_p = \beta - 1$.*

This result is reminiscent of Friedman's argument that the optimal inflation rate is negative (Friedman, 1969). Friedman adopted a money-in-the-utility-function (MIU) framework: a negative rate of inflation provides a return on money holdings sufficient for households to hold the optimum quantity of real balances. Here, the argument is somewhat different. The

[36] In fact we need not assume that the velocity of money is constant: if we allowed for a constant growth rate of the velocity $-1 < g_q \leq 0$, then the inflationary steady state would become $g_q + g_{Mt} = g_{pt} = \hat{g}_p$.

CIA constraint distorts the economy when it binds strictly: when $\lambda_2 > 0$ the labour supply is diminished and output and consumption are below their efficient levels. The optimum inflation rate provides a positive return to holding money which exactly outweighs the effect of discounting and allows for constant consumption without the CIA binding. This removes the distortion induced by the CIA constraint and allows the economy to produce the efficient level of output with the MRS equated to the real wage. Proposition 10 and corollary 7 can be generalised to allow for steady state growth in output and productivity using the conditions in Corollary 5.

4 Capital and Bonds

Thus far, we have abstracted from the presence of capital accumulation and assumed that money is the only asset in the economy. We could introduce capital into our framework by assuming that it is owned by the worker-consumer and rented to the entrepreneurs. Even in the presence of capital, money still contains a savings-based (or precautionary demand) component. In other words, the CIA constraint can be non-binding even in the presence of capital. To show this, let us assume that capital is a factor of the production function which can be written as $x(h_t, k_t; m, \theta_i)$. The extended production function satisfies the usual properties: $x_k > 0$ and $x_{kk} \leq 0$ where x_k and x_{kk} denote the first and second derivatives of $x(\cdot)$ with respect to k. Moreover, we assume that the agents of this economy accumulate capital which depreciates at rate δ. Without loss of generality we also assume that the price of capital is the same as the price of consumption. Then, the euler condition for capital is

$$E_t\left[\left(\frac{\beta u_c(c_{t+1})}{u_c(c_t)}\right)[(1-\delta)+x_k(k_{t+1},\cdot)]\right]= \\ 1-\lambda_{2t}\frac{P_t}{u_c(c_t)}+E_t\left[\left(\frac{\beta\lambda_{2t+1}P_{t+1}}{u_c(c_t)}\right)[(1-\delta)+x_k(k_{t+1},\cdot)]\right] \tag{19}$$

It follows that

$$E_t\left[\left(\frac{\beta u_c\left(c_{t+1}\right)}{u_c\left(c_t\right)}\right)[(1-\delta)+x_k(k_{t+1},\cdot)]\right]\begin{cases} >1 \text{ for } t\in\mathcal{NB} \\ <1 \text{ or } \geq 1 \text{ for } t\in\mathcal{B}\end{cases} \tag{20}$$

while (16) is the corresponding condition for money.[37] Conditions (19) and (20), demonstrate that when there is precautionary demand for money (i.e the CIA constraint does not bind), investment demand is low which means that next period stock of capital is low, and as a result the marginal product of capital is high. Subsequently, the return of capital, measured in utility units, is expected to increase. In this case, condition (19) indicates that the left hand-side of (20) is strictly greater than unity because there is a non-zero possibility that the CIA constraint will bind next period. This demonstrates that even in the presence of capital, money can be used as store of value. If household-consumers knew with absolute certainty that the CIA constraint next period is non-binding (i.e. $\lambda_{2t+1}=0$) then, they would have increased investment demand to the point that the expected utility return of capital equals the expected utility return of money.

Let us consider the case of a non-zero inflation steady state with perfect foresight. If

[37] If capital has a different price than consumption then the left-hand-side of (20) becomes $E_t\left[\left(\frac{\beta u_c(c_{t+1})}{u_c(c_t)}\right)\left[\frac{Q_{t+1}}{Q_t}(1-\delta)+\frac{x_k(k_{t+1},\cdot)}{Q_t}\right]\right]$ where Q_t denotes the relative price of capital (e.g. Cummins and Violante (2002), Fisher (2006)).

we had included capital accumulation, then the return to savings (the marginal return to capital) would be equal to the reciprocal of the discount rate: the optimal inflation rate defined in Proposition 10(ii) would mean that money would have the same rate of return as capital. The steady-state relationship would give a return to capital of

$$(1-\delta)+x_k(k,\cdot)=\frac{1}{\beta}$$

where $x_k(k,\cdot)$ is the steady-state marginal product of capital. The real return to holding one \$ is

$$\frac{1}{1+\hat{g}_p}=\frac{1}{\beta}$$

What would happen if we included interest-bearing nominal assets such as bonds? If we assume the usual arbitrage condition between bonds and capital, these will both offer the same real-return on savings equal to the (expected) marginal return of capital. This will not alter the opportunity cost of holding money from the case of just capital and hence will not eliminate the precautionary-demand for money in the presence of uncertainty. This conclusion depends on how liquid we make Bonds. If we were to make bonds perfectly liquid, then in effect bonds would become an interest bearing form of money and would eliminate the need for non-interest bearing money. Alvarez, Atkeson and Edmond (2009) make an intermediate assumption and allow for bonds to be liquid part of the time and allowing the CIA to be non-binding. Insofar as bonds are not perfectly liquid, there is still a potential role for money over and above the transactions demand.

5 Conclusion

The paper lays out a simple framework in a general equilibrium model with money where the consumption good is produced by monopolistic firms via labor services provided by risk-averse workers. As in Lucas (1982) and Svensson (1985), money is introduced by means of a cash-in-advance constraint. Within this framework, we demonstrate that money is a liquidity vehicle which can have real effects on the economy without requiring the presence of other real assets or any sort of price rigidity. We allow for a very general function for the velocity of money which depends on the current state of the economy. When consumers expect that the value of money next period will decrease below a critical value, they rush to spend all of their money holdings. Then velocity reaches its maximum value and the cash-in-advance constraint binds and we enter a Keynesian world. In this case, both the current variation of money supply as well as the expected value of money affect real variables. When consumers expect that the value of money will not decrease below this critical value, the CIA constraint does not bind as consumers do not spend all their money holdings. In this case we are in a classical world, where real variables are driven only by the current technology innovation and money supply variation affects only the price level. We show that a binding CIA constraint is a welfare inferior outcome for both the workers and firm owners as it delivers lower current utility and lower current real profits for any given level of technology. We also argue that even though the monetary authority can increase the probability of a binding CIA constraint by increasing money supply, expansionary monetary policy can be welfare improving. We also demonstrate that the CIA constraint cannot bind for a lower proportion of the time in a more competitive economy and that when the CIA constraint binds there are cases where prices respond sluggishly to changes in money supply. From a more methodological point of view, we have shown a simple way to introduce monopolistic competition in a general equilibrium monetary model with divisible labor.

References

[1] Adão, B., Correia, I., Teles, P., 2003. Gaps and Triangles. Review of Economic Studies 70, 699-713

[2] Adão, B., Correia, I., Teles, P., 2011. Unique Monetary Equilibria with Interest Rate Rules. Review of Economic Dynamics 14, 432–442

[3] Alvarez, F., Atkeson, A., Edmond, C., 2009. On the Sluggish Response of Prices to Money in an Inventory Theoretic Model of Money Demand. Quarterly Journal of Economics CXXIV (3), 911-967

[4] Benassy, J-P., 1975. NeoKeynesian disequilibrium theory in a monetary economy. Review of Economic Studies 42, 503-23.

[5] Bloise, G., Polemarchakis, H.M., 2006. Theory and Practice of Monetary Policy. Economic Theory 75, 1-23

[6] Chamley, C., Polemarchakis, H., 1984. Assets, General Equilibrium and the Neutrality of Money. Review of Economic Studies 51 (1), 129-138

[7] Chen, H.J., Li, M.C, 2008. Chaotic Dynamics in a Monetary Economy with Habit Persistence. Journal of Economic Behavior and Organization 65, 245–260

[8] Chiu, J., 2007. Endogenously Segmented Asset Market in an Inventory Theoretic Model of Money Demand. Bank of Canada Working Paper 2007-46

[9] Clower, R., 1965. The Keynesian counter-revolution: a theoretical appraisal. Reprinted in Monetary Theory. Ed. R. Clower, 1969, Harmondsworth: Penguin

[10] Cooley, T.F., Hansen, G.D., 1989. The Inflation Tax in a Real Business Cycle Model. American Economic Review 79 (4), 733-748

[11] Cooley, T.F., Hansen, G.D., 1995. Money and the Business Cycle. In Cooley, T., ed., Frontiers of Business Cycle Research. Princeton: Princeton University Press, 175-216

[12] Cooley, T.F., Hansen, G.D., 1997. Unanticipated Money Growth and the Business Cycle Reconsidered. Journal of Money Credit and Banking 29 (4), 624-648

[13] Cummins, J., Violante, G., 2002. Investment-specific technical change in the United States (1947-2000): measurement and macroeconomic consequences. Review of Economic Dynamics 5 (2), 243-284

[14] Datta, B., Dixon, H., 2000. Linear-homothetic preferences. Economics Letters 69, 55-61

[15] Datta, B., Dixon, H., 2001. ω-Homothetic preferences: Theory and applications. Scottish Journal of Political Economy 48 (2), 148-169

[16] Devereux, M., Siu, H., 2007. State dependent Pricing and Business Cycle Asymmetries. International Economic Review 47, 281-310

[17] Díaz-Giméneza, J., Giovannettic, G., Marimon, R., Teles, P., 2008. Nominal Debt as a Burden on Monetary Policy. Review of Economic Dynamics, 11 (3), 493-514

[18] Dotsey, M., Sarte, D.P., 2000. Inflation Uncertainty and Growth in a Cash-in-Advance economy. Journal of Monetary Economics 45, 631-655

[19] Evans, G., Honkapohja, S., Marimon, R., 2007. Stable Sunspot Equilibria in a Cash-in-Advance Economy. B.E Journal of Macroeconomics, Vol. 7: Iss. 1 (Advances), Article 3

[20] Fisher, J., 2006. The dynamic effects of neutral and investment-specific technology shocks. Journal of Political Economy 114 (3), 413-451

[21] Friedman, M., 1969. The Optimum Quantity of Money. In The Optimum Quantity of Money and Other Essays, Chicago: Aldine

[22] Giraud, G., Tsomocos, D., 2010. Nominal Uniqueness and Money Non-neutrality in the Limit-Price Exchange Process. Economic Theory, 45(1), 303-348

[23] Grandmont, J-M, Younes Y., 1972. On the role of money and the existence of monetary equilibrium. Review of Economic Studies, 39, 355-372.

[24] Hromcová, J., 2008. Learning-or-Doing in a Cash-in-Advance Economy with Costly Credit. Journal of Economic Dynamics and Control 32, 2826-2853

[25] Ireland, P., 2003. Implementing the Friedman Rule. Review of Economic Dynamics 6(1), 120-134

[26] Ireland, P., 2005. The liquidity trap, the real balance effect, and the Friedmans rule. International Economic Review 46(4), 1271-1301

[27] Lagos, R., Rocheteau, G., 2005. Inflation, Output, and Welfare. International Economic Review, 46 (2), 495-522

[28] Lagos, R., Wright, R., 2003. Dynamics, Cycles and Sunspot Equilibria in 'Genuinely Dynamic, Fundamentally Disaggregative' Models of Money. Journal of Economic Theory, 109 (2), 156-171

[29] Leijonhufvud, A., 1968. On Keynesian economics and the economics of Keynes. Milton Keynes: Open University Press.

[30] Lucas, R.E.Jr, 1982. Interest Rates and Currency Prices in a Two-Country World. Journal of Monetary Economics 10, 335-359

[31] Malinvaud, E., 1977. The Theory of Unemployment Reconsidered. Oxford: Blackwell.

[32] Patnikin, D., 1956. Money, Interest and Prices: An integration of monetary and value theory. New York, Harper Row

[33] Santos, M., 2006. The value of money in a dynamic equilibrium model. Economic Theory 27, 39-58

[34] Schmitt-Grohé, S., Uribe, M., 2000. Price level determinacy and monetary policy under a balanced-budget requirement. Journal of Monetary Economics 45(1), 211-246

[35] Svensson, L.E.O., 1985. Money and Asset Prices in a Cash-In-Advance Economy. Journal of Political Economy 93 (5), 919-944

[36] Walsh, C., 2003. Monetary theory and policy. 2nd edition, MIT Press

[37] Woodford, M., 2003. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton University Press

Appendix: Proofs

Proof of proposition 1. Suppose the CIA constraint binds. Then, the resource constraint becomes

$$y_t = \underbrace{nm\left[\frac{M_t}{P_t} + \frac{\nu_t}{P_t}\right]}_{c_{WORKERS}} + \underbrace{\frac{1}{\gamma} y_t}_{c_{ENTREPREUNEURS}}$$

which can be rewritten as

$$y_t = \frac{\gamma}{\gamma - 1} nm\left[\frac{M_t}{P_t} + \frac{\nu_t}{P_t}\right] \tag{A.1}$$

and is equivalent to the quantity theory of money equation, $P_t y_t = q^b \overline{M}_t$, where $q^b \equiv \gamma / (\gamma - 1)$.

Next, suppose the CIA constraint does not bind; then, $\lambda_{2t} = 0$. Substituting out $P_t c_t$ from (13) using (11), λ_{1t} from (14) using (13), w_t from (14) using (8) and imposing the equilibrium condition $h_t^s = h_t^d$ we obtain

$$M_{t+1} = \frac{(1+\phi)(\gamma - 1)}{\phi \gamma nm} Y_t - \frac{(\gamma - 1) P_t}{\gamma \phi} \theta_t + [M_t + \nu_t]$$

Using the worker's budget constraint the equilibrium consumption can be written as a linear combination of productivity and real expenditures:

$$c_t = \frac{\gamma - 1}{\gamma \phi} \theta_t - \frac{\gamma - 1}{nm\phi\gamma} y_t$$

It follows that the resource constraint becomes

$$y_t = \underbrace{\frac{nm(\gamma-1)}{\gamma\phi}\theta_t - \frac{(\gamma-1)}{\phi\gamma}y_t}_{C_{WORKERS}} + \underbrace{\frac{1}{\gamma}y_t}_{C_{ENTREPRENEURS}}$$

The latter and the quantity theory of money equation imply

$$y_t = \frac{nm}{1+\phi}\theta_t \quad , \quad c_t = \frac{\gamma-1}{\gamma(1+\phi)}\theta_t \quad \text{and} \quad P_t = (1+\phi)\, q_t^{nb}\left[\frac{M_t+\nu_t}{\theta_t}\right]$$

Then, since $0 < P_t c_t < [M_t + \nu_t]$, it must be the case that

$$0 < \frac{\gamma-1}{\gamma}[M_t+\nu_t]\, q_t^{nb} < [M_t+\nu_t]$$

which holds only if $0 < q_t^{nb} < \gamma/(\gamma-1) \equiv q^b$. ■

Proof of proposition 2. (i) When the CIA contraint strictly binds, equations (8), (9), (13) and (14) imply

$$\lambda_{1t} = \frac{\phi\gamma nm}{(\gamma-1)\, P_t\, [nm\theta_t - y_t]} > 0$$

$$\lambda_{2t} = \frac{(\gamma-1)[nm\theta_t - y_t] - nm\phi\gamma\left[\frac{M_t}{P_t} + \frac{\nu_t}{P_t}\right]}{(\gamma-1)[M_t+\nu_t][nm\theta_t - y_t]} > 0$$

Since $\lambda_{1t} > 0$ and given $(A.1)$, it follows that

$$P_t > q^b\left[\frac{M_t+\nu_t}{\theta_t}\right] \tag{A.2}$$

Likewise, since $\lambda_{2t} > 0$ it follows that

$$P_t > (1+\phi)\, q^b \left[\frac{M_t + \nu_t}{\theta_t}\right] \tag{A.3}$$

Then, $(A.2)$ and $(A.3)$ imply that

$$P_t = (1+\chi_t)\, q^b \left[\frac{M_t + \nu_t}{\theta_t}\right] \text{ where } \chi_t > \phi > 0$$

Using the latter we can express equilibrium real output, real consumption, work effort and real profits as functions of θ_t, χ_t and parameters:

$$y_t = \frac{nm}{1+\chi_t}\theta_t,\ c_t = \frac{\gamma-1}{\gamma nm}y_t,\ \ h_t = \frac{1}{1+\chi_t},\ \ \pi_t = \frac{y_t}{\gamma n}$$

It is straightforward to show that variable χ_t takes a unique value. Recall the euler condition $\lambda_{1t} = \beta E_t \{\lambda_{1t+1} + \lambda_{2t+1}\}$ where

$$\{\lambda_{1t+1} + \lambda_{2t+1}\} = \begin{cases} \frac{q^b}{q_{t+1}} \frac{1}{(M_{t+1}+\nu_{t+1})} & \text{for } (t+1) \in \mathcal{NB} \\ \frac{1}{M_{t+1}+\nu_{t+1}} & \text{for } \ (t+1) \in \mathcal{B} \end{cases}$$

Therefore, given the probability distributions for θ, ν and φ the expectation $E_t \{\lambda_{1t+1} + \lambda_{2t+1}\}$ is well defined. For notational convenience let $Z_t = \beta E_t \{\lambda_{1t+1} + \lambda_{2t+1}\} = \beta E_t\{u_c (c_{t+1}, l_{t+1}) / P_{t+1}\}$. Since

$$\lambda_{1t} = \begin{cases} \frac{\gamma}{(\gamma-1)(M_t+\nu_t)q_t} & \text{for } t \in \mathcal{NB} \\ \frac{\phi}{\chi_t(M_t+\nu_t)} & \text{for } \ t \in \mathcal{B} \end{cases}$$

then

$$\chi_t = \frac{\phi}{Z_t\left(M_t + \nu_t\right)}$$

(ii) From (i), when the CIA constraint is non-binding, $q_t \leq q^b$, equilibrium output and consumption are functions of only θ_t while the price level is a function of M_t, ν_t, φ_t and θ_t. Using the solution for output in (9) and (10), equilibrium work effort and real profits are expressed as functions of only θ_t. Corollary 1 indicates that when the CIA constraint does not bind $Z_t\left(M_t + \nu_t\right) \geq 1$ which implies that $\chi_t \leq \phi$. ■

Proof of corollary 1. The euler condition implies that $Z_t\left(M_t + \nu_t\right) = q^b/q_t$ when the CIA constraint does not bind and $Z_t\left(M_t + \nu_t\right) = \phi/\chi_t$ when the CIA constraint binds. As shown in the proof of proposition 1, when the CIA constraint does not bind $q^b \geq q_t$ and thereby $Z_t\left(M_t + \nu_t\right) \geq 1$. As shown in the proof of proposition 2, when the CIA constraint binds $\chi_t > \phi$ and thereby $Z_t\left(M_t + \nu_t\right) < 1$. ■

Proof of corollary 2. When $t \in \mathcal{NB}$ and the CIA constraint weakly binds $\lambda_{2t} = 0$. Then, the Euler condition becomes $Z_t\left(M_t + \nu_t\right) = 1$ which implies that $\phi = \chi_t$ (see definition of χ_t in proposition 2). Finally, corollary 1 indicates that $q_t = q^b$. ■

Proof of proposition 3. (i) Let $u^{nb}\left(\theta_{t_2}\right) \in \mathcal{U}^{nb} = \{u\left(t\right)\colon\ t \in \mathcal{NB}\}$ and $u^b\left(\theta_{t_1}\right) \in \mathcal{U}^b = \{u\left(t\right)\colon\ t \in \mathcal{B}\}$ correspond to $\chi^{nb}\left(\theta_{t_2}\right)$ and $\chi^b\left(\theta_{t_1}\right)$, respectively. For any θ we know that $\chi^{nb}\left(\theta\right) \leq \phi < \chi^b\left(\theta\right)$. Then, for a given θ, as $\chi^b\left(\theta\right)$ decreses, $\chi^b\left(\theta\right) \to \chi^{nb}\left(\theta\right)$ and $u^b\left(\theta\right) \to u^{nb}\left(\theta\right)$. If $u^b\left(\theta\right)$ increases (decreases) as $\chi^b\left(\theta\right)$ decreases (increases) then $u^{nb}\left(\theta_{t_2}\right) > u^b\left(\theta_{t_1}\right)$. To show this write

$$u^b\left(\theta; \chi^b\right) = \ln \frac{\gamma - 1}{\gamma\left(1 + \chi^b\right)}\theta + \phi \ln \frac{\chi^b}{1 + \chi^b}, \quad \phi < \chi^b$$

Since $0 < \phi < \chi^b$, it follows that

$$\frac{du^b(\theta;\chi)}{d\chi^b} = \frac{\phi - \chi^b}{\chi^b(1+\chi^b)} < 0$$

and thereby $u^{nb}(\theta_{t_2}) > u^b(\theta_{t_1})$. In addition, since $0 < \phi < \chi^b$,

$$\pi^b(\theta_{t_1}) = \frac{m}{\gamma(1+\chi^b)}\theta_{t_1} < \frac{m}{\gamma(1+\phi)}\theta_{t_2} = \pi^{nb}(\theta_{t_2})$$

(ii) From (i), since $du^b(\theta;\chi)/d\chi^b < 0$ and $d\pi^b(\theta;\chi)/d\chi^b < 0$ it follows that for $\theta_{t_1} = \theta_{t_2}$ and $\chi_{t1} > \chi_{t2}$, $u(\theta_{t_2};\chi_{t2}) > u(\theta_{t_1};\chi_{t1})$ and $\pi(\theta_{t_2};\chi_{t2}) > \pi(\theta_{t_1};\chi_{t1})$. ■

Proof of proposition 4. For γ_1 and γ_2 the corresponding upper bounds of velocity are denoted by $q^b(\gamma_1)$ and $q^b(\gamma_2)$, respectively. Proposition 1 indicates that if $\gamma_1 > \gamma_2$ then $q^b(\gamma_1) < q^b(\gamma_2)$. Then, $\mathcal{B}_2(T) \subseteq \mathcal{B}_1(T)$. It follows that $\lim_{T\to\infty} \mathcal{P}(\mathcal{B}_2, T) = \varkappa_2 \leq \lim_{T\to\infty} \mathcal{P}(\mathcal{B}_1, T) = \varkappa_1$. ■

Proof of proposition 5. Recall that $Z_t = \beta E_t\{u_c(c_{t+1}, l_{t+1})/P_{t+1}\} = \beta E_t\{1/P_{t+1}c_{t+1}\}$ where

$$\frac{1}{P_{t+1}c_{t+1}} = \begin{cases} \frac{q^b}{q_{t+1}}\frac{1}{M_{t+1}+\nu_{t+1}} & \text{for } t+1 \in \mathcal{NB} \\ \frac{1}{M_{t+1}+\nu_{t+1}} & \text{for } t+1 \in \mathcal{B} \end{cases}$$

When $t \in \mathcal{NB}$,

$$\frac{P_t c_t}{P_{t+1}c_{t+1}} = \begin{cases} \frac{q_t}{q_{t+1}}\frac{M_t+\nu_t}{M_{t+1}+\nu_{t+1}} & \text{for } t+1 \in \mathcal{NB} \\ \frac{q_t}{q^b}\frac{M_t+\nu_t}{M_{t+1}+\nu_{t+1}} & \text{for } t+1 \in \mathcal{B} \end{cases}$$

and when $t \in \mathcal{B}$,

$$\frac{P_t c_t}{P_{t+1} c_{t+1}} = \begin{cases} \frac{q^b}{q_{t+1}} \frac{M_t + \nu_t}{M_{t+1} + \nu_{t+1}} & \text{for } t+1 \in \mathcal{NB} \\ \frac{M_t + \nu_t}{M_{t+1} + \nu_{t+1}} & \text{for } t+1 \in \mathcal{B} \end{cases}$$

Given θ_t, ν_t and φ_t, and probability distributions ϑ, ξ and $\overline{\Phi}$, if $t(\gamma_1) \in \mathcal{NB}$, it cannot be the case that either $t(\gamma_2) \in \mathcal{NB}$ because $E_t^2 \left[\psi_{t+1} R_{t+1}^M\right] < E_t^1 \left[\psi_{t+1} R_{t+1}^M\right] = 1$ or $t(\gamma_2) \in \mathcal{B}$ because $E_t^2 \left[\psi_{t+1} R_{t+1}^M\right] > 1$. Assuming that ϑ and $\overline{\Phi}$ remain unchanged under both γ_1 and γ_2, the only way that the Euler equation holds is when $t(\gamma_2) \in \mathcal{NB}$ which occurs when the conditional probability distribution ξ has more mass on the left tale. In other words, Ξ^1 first-order stochastically dominates Ξ^2. Likewise, given θ_t, ν_t and φ_t, and probability distributions ϑ, ξ and $\overline{\Phi}$, if $t(\gamma_1) \in \mathcal{B}$, it cannot be the case that either $t(\gamma_2) \in \mathcal{NB}$ because $E_t^2 \left[\psi_{t+1} R_{t+1}^M\right] < 1$ or $t(\gamma_2) \in \mathcal{B}$ because $E_t^2 \left[\psi_{t+1} R_{t+1}^M\right] = \mathit{or} > 1$. Assuming that ϑ and $\overline{\Phi}$ remain unchanged under both γ_1 and γ_2, the only way that the Euler equation holds is when $t(\gamma_2) \in \mathcal{B}$ which occurs when the conditional probability distribution ξ has more mass on the right tale. In other words, Ξ^2 first-order stochastically dominates Ξ^1. ■

Proof of proposition 6. At the steady state the Euler equation, (15), implies that $\beta = \widehat{\lambda}_1/(\widehat{\lambda}_1 + \widehat{\lambda}_2)$. It follows that as long as $\beta \in (0,1)$, $\widehat{\lambda}_2 > 0$ which means that the CIA constraint strictly binds.Using the steady state versions of (8), (13), (14) and (15), the steady state ratio of consumption to leisure can be written as

$$\frac{\widehat{c}}{1-\widehat{h}} = \frac{\gamma - 1}{\gamma \phi (2-\beta)} \widehat{\theta} \tag{A.4}$$

Using the steady state versions of (8) and the wealth constraint (11), the ratio of consumption to work effort can be written as

$$\frac{\widehat{c}}{\widehat{h}} = \frac{\gamma - 1}{\gamma}\widehat{\theta} \tag{A.5}$$

Then, $A.4$ and $A.5$ can be solved for $\widehat{c}$ and $\widehat{h}$:

$$\widehat{h} = \frac{1}{1 + \phi\,(2 - \beta)} \tag{A.6}$$

$$\widehat{c} = \frac{\gamma - 1}{\gamma}\widehat{h\theta} \tag{A.7}$$

It follows that for $\beta \in (0, 1)$

$$\widehat{h} < \frac{1}{1 + \phi} = h^{*}$$

Since $y = nx$ and $\pi = y/n\gamma$, $\widehat{y}$ and $\widehat{\pi}$ can be written as $\widehat{y} = nm\widehat{h\theta}$ and $\widehat{\pi} = \widehat{y}/n\gamma$. Finally, using $A.6$ and $A.7$ along with the quantity theory of money, $\widehat{P}\widehat{y} = \widehat{q}nm\widehat{M}$ and the CIA constraint, $\widehat{P} = \widehat{M}/\widehat{c}$, the steady state price level can be written as:

$$\widehat{P} = [1 + \phi\,(2 - \beta)]\widehat{q}\left[\frac{\widehat{M}}{\widehat{\theta}}\right] \text{ where } \widehat{q} = q^{b}$$

■

Proof of proposition 7. We prove (i) and (ii) simultaneously. From (16), $t \in \mathcal{B}$ means that

$$\frac{P_t c_t}{P_{t+1} c_{t+1}} < \frac{1}{\beta} \tag{A.8}$$

where $P_t c_t = M_t + \nu_t$. There are two possible cases for $t+1$: (1) $t+1 \in \mathcal{B}$ and (2) $t+1 \in \mathcal{NB}$.

(1) When $t+1 \in \mathcal{B}$,

$$\frac{M_t + \nu_t}{M_{t+1} + \nu_{t+1}} = \frac{1}{\left(M_{t+1} + \nu_{t+1}\right) / \left(M_t + \nu_t\right)} = \frac{1}{M_{t+2}/M_{t+1}} = \frac{1}{1 + g_{Mt+2}} < \frac{1}{\beta}$$

or $g_{Mt+2} > \beta - 1$. Therefore, (i) when $t \in \mathcal{B}$ and $t+1 \in \mathcal{B}$ then $g_{Mt+2} > \beta - 1$ and (ii) if $t+1 \in \mathcal{B}$ and $g_{Mt+2} \leq \beta - 1$ then $t \in \mathcal{NB}$.

(2) When $t+1 \in \mathcal{NB}$, $P_{t+1}c_{t+1} = \left(q_{t+1}/q^b\right)\left(M_{t+1} + \nu_{t+1}\right)$ where $q_{t+1} \leq q^b$. Then, $A.8$ implies

$$\frac{M_t + \nu_t}{M_{t+1} + \nu_{t+1}} \frac{q^b}{q_{t+1}} = \frac{1}{1 + g_{Mt+2}} \frac{q^b}{q_{t+1}} < \frac{1}{\beta}$$

Since $q_{t+1} \leq q^b$

$$\frac{1}{1 + g_{Mt+2}} < \frac{1}{\beta}$$

or $g_{Mt+2} > \beta - 1$. Therefore, (i) when $t \in \mathcal{B}$ and $t+1 \in \mathcal{NB}$ then $g_{Mt+2} > \beta - 1$ and (ii) if $t+1 \in \mathcal{NB}$ and $g_{Mt+2} \leq \beta - 1$ then $t \in \mathcal{NB}$.

From (1) and (2), it follows that (i) when $t \in \mathcal{B}$ then $g_{Mt+2} > \beta - 1$ and (ii) when $g_{Mt+2} \leq \beta - 1$ then $t \in \mathcal{NB}$. Since

$$g_{Mt+2} = (1 + g_{vt+1})(1 - \frac{M_t}{M_t + v_t})$$

conditions $g_{Mt+2} > \beta - 1$ and $g_{Mt+2} \leq \beta - 1$ can be written as $g_{\nu t+1} > [(M_t + \nu_t)/\nu_t](\beta - 1) - 1$ and $g_{\nu t+1} \leq [(M_t + \nu_t)/\nu_t](\beta - 1) - 1$, respectively. These conditions can also be written as $\nu_{t+1} > (M_t + \nu_t)(\beta - 1)$ and $\nu_{t+1} \leq (M_t + \nu_t)(\beta - 1)$. What is left is to show that (i) $g_{Mt+2} > \beta - 1$ does not always imply that $t \in \mathcal{B}$, and (ii) $t \in \mathcal{NB}$ does not always imply that

$g_{Mt+2} \leq \beta - 1$: (i) It is enough to find a case where for $g_{Mt+2} > \beta - 1$, $t \in \mathcal{NB}$. Condition $g_{Mt+2} + 1 > \beta$ can be written as

$$\frac{M_t + \nu_t}{M_{t+1} + \nu_{t+1}} > \frac{1}{\beta} \tag{A.9}$$

when $t \in \mathcal{NB}$ and $t + 1 \in \mathcal{NB}$ then, the ratio of consumption expenditures between period t and period $t + 1$ can be written as

$$\frac{P_t c_t}{P_{t+1} c_{t+1}} = \frac{\left(q_t / q^b\right)\left(M_t + \nu_t\right)}{\left(q_{t+1} / q^b\right)\left(M_{t+1} + \nu_{t+1}\right)} = \frac{1}{\beta} \tag{A.10}$$

which is the state of condition (16) when the CIA constraint does not bind. Condition $A.9$ is consistent with condition $A.10$ when $q_{t+1} > q_t$. Since the latter is possible we found a case where $g_{Mt+2} > \beta - 1$ does not imply $t \in \mathcal{B}$. (ii) Likewise, to show that $t \in \mathcal{NB}$ does not always imply that $g_{Mt+2} \leq \beta - 1$, it is enough to find a case where for $t \in \mathcal{NB}$, $g_{Mt+2} > \beta - 1$. If $t \in \mathcal{NB}$ and $t + 1 \in \mathcal{NB}$ then (16) becomes $A.10$. If $q_{t+1} < q_t$ then this implies that

$$\frac{M_t + \nu_t}{M_{t+1} + \nu_{t+1}} < \frac{1}{\beta}$$

which is equivalent to $g_{Mt+2} > \beta - 1$. Since this is possible, we found a case where $t \in \mathcal{NB}$ does not imply that $g_{Mt+2} \leq \beta - 1$. ■

Proof of corollary 3. For any $T \in \mathbb{Z}_+ \cup \{0\}$ and $\beta \in (0, 1)$, $\mathcal{M}^{>}(T) = \varnothing$ as long as for all $t \geq T + 1$, $-1 < g_{Mt+2} \leq \beta - 1$. For any $T \in \mathbb{Z}_+ \cup \{0\}$ and $\beta \in (0, 1)$, it is perfectly possible that $\mathcal{M}^{\leq}(T) = \varnothing$ since $M_t > 0$ for all $t \geq T + 1$. Therefore, for $g_{Mt+2} > -1$ with

$t \geq T+1$, $\varnothing \subseteq \mathcal{M}^{\leq}(T)$ and $\varnothing \subseteq \mathcal{M}^{>}(T)$. Let $\mathbb{Z}_{+}(T) = \{T+1, T+2, ..\infty\}$ such that $\mathbb{Z}_{+}(T) = \mathcal{M}^{\leq}(T) \cup \mathcal{M}^{>}(T)$. Note that (i) if $\widetilde{\mathcal{B}}(T) \subseteq \mathbb{Z}_{+}(T)$ then $\varnothing \subseteq \widetilde{\mathcal{NB}}(T)$ and (ii) if $\widetilde{\mathcal{NB}}(T) \subseteq \mathbb{Z}_{+}(T)$ then $\varnothing \subseteq \widetilde{\mathcal{B}}(T)$: (i) This is trivial since proposition 6 and its proof indicate that $\widetilde{\mathcal{NB}}(T) \subseteq \mathbb{Z}_{+}(T)$ which implies $\varnothing \subseteq \widetilde{\mathcal{B}}(T)$. (ii) From the fact that $\varnothing \subseteq \mathcal{M}^{\leq}(T)$ and proposition 6 we know that $\widetilde{\mathcal{B}}(T) \subseteq \mathbb{Z}_{+}(T)$ which implies $\varnothing \subseteq \widetilde{\mathcal{NB}}(T)$. Notice that the zero-inflation steady state is a case where $\mathcal{NB} = \varnothing$. ■

Proof of proposition 8. The only difference between the equilibrium of the economy with certainty and the equilibrium of the economy with uncertainty is the fact that in the economy with uncertainty χ_t holds in expectation. As shown in the proof of proposition 2, the Euler equation becomes

$$\frac{\phi}{\chi_t\left(M_t+\nu_t\right)} = \begin{cases} \frac{\beta}{(M_{t+1}+\nu_{t+1})} \frac{q^b}{q_{t+1}} \text{ for } t+1 \in \mathcal{NB} \\ \frac{\beta}{M_{t+1}+\nu_{t+1}} \text{ for } t+1 \in \mathcal{B} \end{cases} \tag{A.11}$$

Then, $A.11$ can be solved for χ_t. ■

Proof of corollary 4. Using proposition 7, for any $t \in \mathcal{NB}$

$$(1+g_{ct+1}) = \begin{cases} (1+g_{\theta t+1}) & \text{for } t+1 \in \mathcal{NB} \\ (1+g_{\theta t+1})\,\frac{1+\phi}{1+\chi_{t+1}} & \text{for } t+1 \in \mathcal{B} \end{cases}$$

Since $\chi_{t+1} > \phi$ for $t+1 \in \mathcal{B}$ (proposition 8), then for any $t \in \mathcal{NB}$, $g_{ct+1} \leq g_{\theta t+1}$. ■

Proof of proposition 9. (i) For any $t \in \mathcal{NB}$, the euler equation becomes $(1+g_{pt+1})(1+g_{ct+1}) = \beta$. Then, from corollary 4 we know that when $t \in \mathcal{NB}$ then

$$g_{pt+1} \geq \frac{\beta}{1+g_{\theta t+1}} - 1 \tag{A.12}$$

However, the reverse does not always hold: when $t \in \mathcal{B}$ and $t+1 \in \mathcal{NB}$ then

$$1 + g_{ct+1} = (1 + g_{\theta t+1}) \frac{1+\chi_t}{1+\phi}$$

where $\chi_t > \phi$. The latter implies that $g_{ct+1} > g_{\theta t+1}$. Then from the euler equation we know that if $t \in \mathcal{B}$ and $t+1 \in \mathcal{NB}$ then $A.12$ holds with strict inequality.When $t \in \mathcal{B}$ and $t+1 \in \mathcal{B}$,

$$1 + g_{ct+1} = (1 + g_{\theta t+1}) \frac{1+\chi_t}{1+\chi_{t+1}}$$

Then for $t \in \mathcal{B}$ and $t+1 \in \mathcal{B}$, $\chi_t < \chi_{t+1} \Longrightarrow g_{ct+1} < g_{\theta t+1}$ and $\chi_t \geq \chi_{t+1} \Longrightarrow g_{ct+1} \geq g_{\theta t+1}$. Thus, for $t \in \mathcal{B}$ and $t+1 \in \mathcal{B}$ we can find χ_t and χ_{t+1} such that $\chi_t > \chi_{t+1}$ so that $A.12$ holds with equality.

(ii) We have established that for any $t \in \mathcal{NB}$, $A.12$ holds. The latter implies that when

$$g_{pt+1} < \frac{\beta}{1+g_{\theta t+1}} - 1 \tag{A.13}$$

then $t \in \mathcal{B}$. However, $t \in \mathcal{B}$ does not always imply $A.13$. As shown above, for $t \in \mathcal{B}$ and $t+1 \in \mathcal{B}$ we can find χ_t and χ_{t+1} such that $\chi_t > \chi_{t+1}$ so that $A.12$ holds with equality. In addition, for $t \in \mathcal{B}$, $t+1 \in \mathcal{B}$ and $\chi_t < \chi_{t+1}$, $A.12$ holds with strict inequality. ■

Proof of corollary 5. Condition (16) indicates that $(t-1) \in \mathcal{NB}$ if and only if

$$g_{pt} = \frac{\beta}{1+g_{ct}} - 1 \tag{A.14}$$

otherwise $(t-1) \in \mathcal{B}$ and

$$g_{pt} > \frac{\beta}{1+g_{ct}} - 1 \tag{A.15}$$

(i) When $(t-1) \in \mathcal{NB}$ and $t \in \mathcal{NB}$ then, $g_{ct} = g_{\theta t}$ and $A.14$ becomes

$$g_{pt} = \frac{\beta}{1+g_{\theta t}} - 1 \tag{A.16}$$

To show that the reverse does not always hold, it is enough to find a case where $A.16$ holds and either $t-1$ or t or both $\notin \mathcal{NB}$. Suppose that both $t-1$ and $t \in \mathcal{B}$. Then, $A.15$ becomes

$$g_{pt} > \frac{\beta}{1+g_{\theta t}}(1+g_{\chi t}) - 1 \tag{A.17}$$

In this case, $A.16$ is consistent with $A.17$ as long as $g_{\chi t} < 0$ which is feasible.

(ii) If $(t-1) \in \mathcal{NB}$ and $t \in \mathcal{B}$ then $1 + g_{ct} = (1+g_{\theta t})(1+\phi)/(1+\chi_t)$. Using the latter in $A.14$, we obtain

$$g_{pt} = \frac{\beta}{1+g_{\theta t}} \frac{1+\chi_t}{1+\phi} - 1 \tag{A.18}$$

Since $\chi_t > \phi$, $A.18$ implies $g_{pt} > [\beta/(1+g_{\theta t})] - 1$. To show that the reverse does not always hold, it is enough to find a case where $A.18$ holds and either both $t-1$ and $t \in \mathcal{B}$ or $t-1 \in \mathcal{B}$

and $t \in \mathcal{NB}$. Suppose that both $t-1$ and $t \in \mathcal{B}$. Then, $A.18$ is consistent with $A.17$ as long as $g_{\chi t} < [(1+\chi_t)/(1+\phi)] - 1$ which is feasible.

(iii) If $(t-1) \in \mathcal{B}$ and $t \in \mathcal{B}$ then,

$$g_{pt} > \frac{\beta}{1+g_{\theta t}} \frac{1+\chi_{t-1}}{1+\chi_t} - 1$$

which implies that either $g_{pt} > [\beta/(1+g_{\theta t})] - 1$ or $g_{pt} \leq [\beta/(1+g_{\theta t})] - 1$ depending on the value of $(1+\chi_{t-1})/(1+\chi_t)$. Likewise, if $(t-1) \in \mathcal{B}$ and $t \in \mathcal{NB}$ then,

$$g_{pt} > \frac{\beta}{1+g_{\theta t}} \frac{1+\phi}{1+\chi_{t-1}} - 1$$

which implies that either $g_{pt} > [\beta/(1+g_{\theta t})] - 1$ or $g_{pt} \leq [\beta/(1+g_{\theta t})] - 1$ depending on the value of $(1+\phi)/(1+\chi_{t-1})$. ■

Proof of corollary 6. (i) If $(t-1) \in \mathcal{NB}$ and $t \in \mathcal{NB}$, Proposition 8 and corollary 5 (i) indicate that

$$1 + g_{pt} = \frac{(1+g_{qt})(1+g_{Mt+1})}{(1+g_{\theta t})} \tag{A.19}$$

and $1 + g_{pt} = [\beta/(1+g_{\theta t})]$, respectively. Combining the two, $A.19$ reduces to

$$g_{Mt+1} = \frac{\beta}{1+g_{qt}} - 1 \tag{A.20}$$

However, $A.20$ does not always imply that $(t-1) \in \mathcal{NB}$ and $t \in \mathcal{NB}$. Suppose that $(t-1) \in \mathcal{NB}$ and $t \in \mathcal{B}$. Then, proposition 8 and corollary 5 (ii) indicate that

$$1 + g_{pt} = \frac{(1 + g_{qt})(1 + g_{Mt+1})}{(1 + g_{\theta t})} \frac{1 + \chi_t}{1 + \phi} \text{ with } \chi_t > \phi \tag{A.21}$$

and $1 + g_{pt} > [\beta/(1 + g_{\theta t})]$, respectively. Using the latter in $A.21$, we obtain $1 + g_{Mt+1} > [\beta/(1 + g_{\theta t})][(1 + \phi)/(1 + \chi_t)]$.Since $\chi_t > \phi$, it could be the case that $A.20$ holds.

(ii) If $(t-1) \in \mathcal{NB}$ and $t \in \mathcal{B}$, it is shown in (i) that either $g_{Mt+1} > [\beta/(1 + g_{\theta t})] - 1$ or $g_{Mt+1} \leq [\beta/(1 + g_{\theta t})] - 1$, depending on the value of $(1 + \phi)/(1 + \chi_t)$. Proposition 8 and corollary 5 (iii), imply that if $(t-1) \in \mathcal{B}$ then, $g_{Mt+1} > [\beta/(1 + g_{\theta t})] - 1$ or $g_{Mt+1} \leq [\beta/(1 + g_{\theta t})] - 1$ for any t. ■

Proof of Proposition 10. (i) and (ii) follow from Proposition 7 and corrolary 4 under the assumption of an inflationary steady-state. To establish (iii), note that (given $g_\theta = 0$) $g_c > 0$ if $g_p < \beta - 1$, hence no steady-state with $g_c = 0$ exists. ■

Proof of corollary 7. It follows from Propositions 3(i) and 10(ii). ■